

Extended Version: Refinement Reflection: Complete Verification with SMT

NIKI VAZOU, University of Maryland

ANISH TONDWALKAR, RANJIT JHALA, University of California, San Diego

VIKRAMAN CHOUDHURY, RYAN G. SCOTT, RYAN R. NEWTON, Indiana University

PHILIP WADLER, University of Edinburgh

We introduce *Refinement Reflection*, a new framework for building SMT-based deductive verifiers. The key idea is to reflect the code implementing a user-defined function into the function's (output) refinement type. As a consequence, at uses of the function, the function definition is instantiated in a precise fashion that permits decidable verification. We show how reflection allows the user to write *equational proofs* of programs just by writing other programs *e.g.* using pattern-matching and recursion to perform case-splitting and induction. Thus, via, the propositions-as-types principle we show that reflection permits the *specification* of arbitrary functional correctness properties. While equational proofs are easy, writing them out can be exhausting. We introduce a proof-search algorithm called *Proof by Logical Evaluation* that uses techniques from model checking & abstract interpretation, to completely automate equational reasoning. We have implemented reflection in LIQUID HASKELL and used it to verify that the widely used instances of the Monoid, Applicative, Functor, and Monad typeclasses actually satisfy key algebraic laws required to make the clients safe, and to build the first library that actually verifies assumptions about associativity and ordering that are crucial for safe deterministic parallelism.

1 INTRODUCTION

Deductive verifiers fall roughly into two camps. Satisfiability Modulo Theory (SMT) based verifiers (*e.g.* DAFNY and F*) use fast decision procedures to completely automate the verification of programs that only require reasoning over a fixed set of theories like linear arithmetic, string, set and bitvector operations. These verifiers, however, encode the semantics of user-defined functions with universally-quantified axioms and use incomplete (albeit effective) heuristics to instantiate those axioms. These heuristics make it difficult to characterize the kinds of proofs that can be automated, and hence, explain why a given proof attempt fails [Leino and Pit-Claudel 2016]. At the other end, we have Type-Theory (TT) based theorem provers (*e.g.* COQ and AGDA) that use type-level computation (normalization) to facilitate principled reasoning about terminating user-defined functions, but which require the user to supply lemmas or rewrite hints to discharge proofs over decidable theories.

We introduce *Refinement Reflection*, a new framework for building SMT-based deductive verifiers, which permits the specification of arbitrary properties and yet enables complete, automated SMT-based reasoning about user-defined functions. In previous work, refinement types [Constable and Smith 1987; Rushby et al. 1998] – which decorate basic types (*e.g.* `Integer`) with SMT-decidable predicates (*e.g.* $\{v: \text{Integer} \mid 0 \leq v \ \&\& \ v < 100\}$) – were used to retrofit so-called shallow verification, such as array bounds checking, into several languages: ML [Bengtson et al. 2008; Rondon et al. 2008; Xi and Pfenning 1998], C [Condit et al. 2007; Rondon et al. 2010], Haskell [Vazou et al. 2014a], TypeScript [Vekris et al. 2016], and Racket [Kent et al. 2016].

1. Refinement Reflection Our first contribution is the notion of *refinement reflection*. To reason about user-defined functions, the function's implementation can be *reflected* into its (output)

2017. 2475-1421/2017/10-ART1 \$15.00

<https://doi.org/10.1145/nnnnnnn.nnnnnnn>

refinement-type specification, thus converting the function's type signature into a precise description of the function's behavior. This simple idea has a profound consequence: at *uses* of the function, the standard rule for (dependent) function application yields a precise means of reasoning about the function (§ 4).

2. Complete Specification Our second contribution is a *library of combinators* that lets programmers compose sophisticated *proofs* from basic refinements and function definitions. Our proof combinators let programmers use existing language mechanisms like branches (to encode case splits), recursion (to encode induction), and functions (to encode auxiliary lemmas) to write proofs that look very much like their pencil-and-paper analogues (§ 2). Furthermore, since proofs are literally just programs, we use the principle of propositions-as-types [Wadler 2015] (known as Curry-Howard isomorphism [Howard 1980]) to show, via a recipe for encoding proofs with alternating quantifiers, that SMT-based verifiers can express any natural deduction proof, and provide a pleasant implementation of natural deduction that can be used for pedagogical purposes (§ 3).

3. Complete Verification While equational proofs can be very easy and expressive, writing them out can quickly get exhausting. Our third contribution is *Proof by Logical Evaluation* (PLE) a new proof-search algorithm that completely automates equational reasoning. The key idea in PLE is to mimic type-level computation within SMT-logics by representing functions in a *guarded form* [Dijkstra 1975] and repeatedly unfolding function application terms by instantiating them with their definition corresponding to an *enabled* guard. We formalize a notion of equational proof and show that the above strategy is *complete*: *i.e.* it is guaranteed to find an equational proof if one exists. Furthermore, using techniques from the literature on Abstract Interpretation [Cousot and Cousot 1977] and Model Checking [Clarke et al. 1992], we show that the above proof search corresponds to a *universal* (or *must*) abstraction of the concrete semantics of the user-defined functions. Thus, as those functions are total we obtain the pleasing guarantee that proof search terminates (§ 6).

We evaluate our approach by implementing refinement reflection and PLE in LIQUID HASKELL [Vazou et al. 2014a], thereby turning Haskell into a theorem prover. Repurposing an existing programming language allows us to take advantage of a mature compiler and an ecosystem of libraries, while keeping proofs and programs in the same language. We demonstrate the benefits of this conversion by proving typeclass laws. Haskell's typeclass machinery has led to a suite of expressive abstractions and optimizations which, for correctness, crucially require typeclass *instances* to obey key algebraic laws. We show how reflection and PLE can be used to verify that widely used instances of the Monoid, Applicative, Functor, and Monad typeclasses satisfy the respective laws. Finally, we use reflection to create the first deterministic parallelism library that actually verifies assumptions about associativity and ordering that ensure determinism (§ 7).

Thus, our results demonstrate that Refinement Reflection and Proof by Logical Evaluation identify a new design for deductive verifiers which, by combining the complementary strengths of SMT- and TT- based approaches, enables complete verification of expressive specifications spanning decidable theories and user defined functions.

2 OVERVIEW

We start with an overview of how SMT-based refinement reflection lets us write proofs as plain functions and how PLE automates equational reasoning.

2.1 Refinement Types

First, we recall some preliminaries about specification and verification with refinement types.

Refinement types are the source program's (here Haskell's) types refined with logical predicates drawn from an SMT-decidable logic [Constable and Smith 1987; Rushby et al. 1998]. For example, we define `Nat` as the set of `Integer` values v that satisfy the predicate $0 \leq v$ from the quantifier-free logic of linear arithmetic and uninterpreted functions (QF-UFLIA [Barrett et al. 2010]):

```
type Nat = { v:Integer | 0 ≤ v }
```

Specification & Verification Throughout this section, to demonstrate the proof features we add to LIQUID HASKELL, we will use the textbook Fibonacci function which we type as follows.

```
fib :: Nat → Nat
fib 0 = 0
fib 1 = 1
fib n = fib (n-1) + fib (n-2)
```

To ensure termination, the input type's refinement specifies a *pre-condition* that the parameter must be `Nat`. The output type's refinement specifies a *post-condition* that the result is also a `Nat`. Refinement type checking can automatically verify that if `fib` is invoked with a non-negative `Integer`, then it terminates and yields a non-negative `Integer`.

Propositions We can define a data type representing propositions as an alias for `unit`:

```
type Prop = ()
```

which can be *refined* with propositions about the code, e.g. that $2 + 2$ equals 4

```
type Plus_2_2 = { v: Prop | 2 + 2 = 4 }
```

For simplicity, in LIQUID HASKELL, we abbreviate the above to `type Plus_2_2 = { 2 + 2 = 4 }`.

Universal & Existential Propositions Using the standard encoding of Howard [1980], known as Curry-Howard isomorphism, refinements encode universally-quantified propositions as *dependent function* types of the form:

```
type Plus_comm = x:Integer → y:Integer → { x + y = y + x }
```

As x and y refer to arbitrary inputs, any inhabitant of the above type is a proof that `Integer` addition commutes.

Refinements encode existential quantification via *dependent pairs* of the form:

```
type Int_up = n:Integer → (m::Integer, {n < m})
```

The notation $(m :: t, t')$ describes dependent pairs where the name m for the first element can appear inside refinements for the second element. Thus, `Int_up` states the proposition that for every integer n , *there exists* one that is larger than n .

While quantifiers cannot appear directly inside the refinements, dependent functions and pairs allow us to specify quantified propositions. One limitation of this encoding is that quantifiers cannot exist inside refinement's logical connectives (like \wedge and \vee). In this paper, we present how to encode logical connectives using data types, e.g. conjunction as product and disjunction as a union, and show how to specify arbitrary, quantified propositions using refinement types, i.e. have complete specifications, and how to verify those propositions using refinement type checking (§ 3).

Proofs We *prove* the above propositions by writing Haskell programs, for example

```
plus_2_2 :: Plus_2_2   plus_comm :: Plus_comm   int_up :: Int_up
plus_2_2 = ()         plus_comm = \x y → ()   int_up = \n → (n+1, ())
```

Standard refinement typing reduces the above to the respective *verification conditions* (VCs)

$$true \Rightarrow 2 + 2 = 4 \quad \forall x, y. true \Rightarrow x + y = y + x \quad \forall n. n < n + 1$$

which are easily deemed valid by the SMT solver, allowing us to prove the respective propositions.

A Note on Bottom: Readers familiar with Haskell’s semantics may be concerned that “bottom”, which inhabits all types, makes our proofs suspect. Fortunately, as described in Vazou et al. [2014a], LIQUID HASKELL ensures that all terms with non-trivial refinements provably terminate and evaluate to (non-bottom) values, which makes our proofs sound.

2.2 Refinement Reflection

Suppose we wish to prove properties about the `fib` function, e.g. that $\{\text{fib } 2 = 1\}$. Standard refinement type checking runs into two problems. First, for decidability and soundness, *arbitrary* user-defined functions cannot belong in the refinement logic, i.e. we cannot *refer* to `fib` in a refinement. Second, the only specification that a refinement type checker has about `fib` is its type `Nat → Nat` which is too weak to verify $\{\text{fib } 2 = 1\}$. To address both problems, we **reflect** `fib` into the logic which sets the three steps of refinement reflection in motion.

Step 1: Definition The annotation creates an *uninterpreted function* `fib :: Integer → Integer` in the refinement logic. By uninterpreted, we mean that the logical `fib` is *not* connected to the program function `fib`; in the logic, `fib` only satisfies the *congruence axiom* $\forall n, m. n = m \Rightarrow \text{fib } n = \text{fib } m$. On its own, the uninterpreted function is not terribly useful: we cannot check $\{\text{fib } 2 = 1\}$ as the SMT solver *cannot* prove the VC $true \Rightarrow \text{fib } 2 = 1$ which requires reasoning about `fib`’s *definition*.

Step 2: Reflection In the next key step, we reflect the *definition* of `fib` into its refinement type by automatically strengthening the user defined type for `fib` to:

```
fib :: n:Nat → { v:Nat | v = fib n && fibP n }
```

where `fibP` is an alias for a refinement *automatically derived* from the function’s definition:

```
fibP n = n == 0 ⇒ fib n = 0
      ∧ n == 1 ⇒ fib n = 1
      ∧ n >= 1 ⇒ fib n = fib (n-1) + fib (n-2)
```

Step 3: Application With the reflected refinement type, each application of `fib` in the code automatically *unfolds* the definition of `fib` *once* in the logic. We prove $\{\text{fib } 2 = 1\}$ by:

```
pf_fib2 :: { fib 2 = 1 }
pf_fib2 = let { t0 = fib 0; t1 = fib 1; t2 = fib 2 } in ()
```

We write in bold red, **f**, to highlight places where the unfolding of `f`’s definition is important. Via refinement typing, the above yields the following VC that is discharged by SMT, even though `fib` is uninterpreted:

$$((\text{fibP } 0) \wedge (\text{fibP } 1) \wedge (\text{fibP } 2)) \Rightarrow (\text{fib } 2 = 1)$$

Note that the verification of `pf_fib2` relies merely on the fact that `fib` is applied to (i.e. unfolded at) 0, 1 and 2. The SMT solver automatically *combines* the facts, once they are in the antecedent. The following is also verified:

```
pf_fib2' :: {v:[Nat] | fib 2 = 1 }
pf_fib2' = [ fib 0, fib 1, fib 2 ]
```

In the next subsection, we will continue to use explicit, step-by-step proofs as above, but we introduce additional tools for proof composition. Then, in § 2.4 we will eliminate unnecessary details in such proofs, using *Proof by Logical Evaluation* (PLE) for automation.

2.3 Equational Proofs

We can structure proofs to follow the style of *calculational* or *equational* reasoning popularized in classic texts [Bird 1989; Dijkstra 1976] and implemented in AGDA [Mu et al. 2009] and DAFNY [Leino and Polikarpova 2016]. To this end, we have developed a library of proof combinators that permits reasoning about equalities and linear arithmetic.

“Equation” Combinators We equip LIQUID HASSELL with a family of equation combinators, \odot , for logical operators in the theory QF-UFLIA, $\odot \in \{=, \neq, \leq, <, \geq, >\}$. (In Haskell code, to avoid collisions with existing operators, we further append a period “.” to these operators, so that “=” becomes “=.” instead.) The refinement type of \odot *requires* that $x \odot y$ holds and then *ensures* that the returned value is equal to x . For example, we define =. as:

```
(=.) :: x:a → y:{ a | x = y} → { v:a | v = x }
x =. _ = x
```

and use it to write the following “equational” proof:

```
fib2_1 :: { fib 2 = 1 }
fib2_1 = fib 2 =. fib 1 + fib 0 =. 1 ** QED
```

where **** QED** constructs “proof terms” by “casting” expressions to **Prop** in a post-fix fashion.

```
data QED = QED          (**) :: a → QED → Prop
_ ** QED = ()
```

“Because” Combinators Often, we need to compose lemmas into larger theorems. For example, to prove `fib 3 = 2` we may wish to reuse `fib2_1` as a lemma. We do so with a “because” combinator:

```
(∴) :: (Prop → a) → Prop → a
f ∴ y = f y
```

The operator is simply an alias for function application that lets us write $x \odot y \ ∴ \ p$. We use the because combinator to prove that `fib 3 = 2`.

```
fib3_2 :: { fib 3 = 2 }
fib3_2 = fib 3 =. fib 2 + fib 1 =. 2 ∴ fib2_1 ** QED
```

Here `fib 2` is not important to unfold, because `fib2_1` already provides the same information.

Arithmetic and Ordering Next, let’s see how we can use arithmetic and ordering to prove that `fib` is (locally) increasing, *i.e.* for all n , `fib n ≤ fib (n + 1)`.

```
type Up f = n:Nat → {f n ≤ f (n + 1)}

fibUp  :: Up fib
fibUp 0 = fib 0 <. fib 1          ** QED
fibUp 1 = fib 1 ≤. fib 1 + fib 0  =. fib 2    ** QED
fibUp n = fib n ≤. fib n + fib (n-1) =. fib (n+1) ** QED
```

Case Splitting The proof `fibUp` works by splitting cases on the value of n . In the *base* cases 0 and 1, we simply assert the relevant inequalities. These are verified as the reflected refinement unfolds the definition of `fib` at those inputs. The derived VCs are (automatically) proved as the SMT solver concludes $0 < 1$ and $1 + 0 \leq 1$ respectively. When n is greater than two, `fib n` is unfolded to `fib (n-1) + fib (n-2)`, which, as `fib (n-2)` is non-negative, completes the proof.

Induction & Higher Order Reasoning Refinement reflection smoothly accomodates induction and higher-order reasoning. For example, lets prove that every function f that increases locally (*i.e.* $f\ z \leq f\ (z+1)$ for all z) also increases globally (*i.e.* $f\ x \leq f\ y$ for all $x < y$)

```

type Mono = f : (Nat → Integer) → Up f → x : _ → y : {x < y} → {f x ≤ f y}

fMono :: Mono / [y]
fMono f up x y
  | x+1 == y = f x ≤. f (x+1) ∴ up x ≤. f y                ** QED
  | x+1 < y = f x ≤. f (y-1) ∴ fMono f up x (y-1) ≤. f y ∴ up (y-1) ** QED

```

We prove the theorem by induction on y as specified by the annotation $/ [y]$ which states that y is a well-founded termination metric that decreases at each recursive call [Vazou et al. 2014a]. If $x+1 == y$, then we call the $up\ x$ proof argument. Otherwise, $x+1 < y$, and we use the induction hypothesis *i.e.* apply $fMono$ at $y-1$, after which transitivity of the less-than ordering finishes the proof. We can *apply* the general $fMono$ theorem to prove that `fib` increases monotonically:

```

fibMono :: n:Nat → m:{n < m} → {fib n ≤ fib m}
fibMono = fMono fib fibUp

```

2.4 Complete Verification: Automating Equational Reasoning

While equational proofs can be very easy, writing them out can quickly get exhausting. Lets face it: `fib3_2` is doing rather a lot of work just to prove that `fib 3` equals `2!` Happily, the *calculational* nature of such proofs allows us to develop the following proof search algorithm PLE that is inspired by model checking [Clarke et al. 1992]:

- **Step 1: Guard Normal Form** First, as shown in the definition of `fibP` above, each reflected function is transformed into a *guard normal form* $\wedge_i(p_i \Rightarrow f(\bar{x}) = b_i)$ *i.e.* as a collection of *guards* p_i and their corresponding definition b_i .
- **Step 2: Unfolding** Second, given a VC of the form $\Phi \Rightarrow p$, we iteratively *unfold* function application terms in Φ and p by *instantiating* them with the definition corresponding to an *enabled* guard, where we check enabled-ness by querying the SMT solver. For example, given a VC $true \Rightarrow \text{fib } 3 = 2$, the guard $3 \geq 1$ is trivially *enabled*, *i.e.* is true, and hence we strengthen the hypothesis Φ with the equality $\text{fib } 3 = \text{fib } 3 - 1 + \text{fib } 3 - 2$ corresponding to unfolding the definition of `fib` at 3.
- **Step 3: Fixpoint** Third, we repeat the above process until either the goal is proved or we have reached a fixpoint, *i.e.* no further unfolding is enabled. For example, the above fixpoint computation unfolds the definition of `fib` at 3, 2, 1, and 0 and then stops as no further guards are enabled.

Automatic Equational Reasoning In § 6 we formalize a notion of *equational proof* and show that the proof search procedure PLE enjoys two key properties. First, that it is guaranteed to find an equational proof if one exists. Second, that under certain conditions readily met in practice, it is guaranteed to terminate. These two properties allow us to use PLE to predictably automate proofs: the programmer needs *only* supply the relevant induction hypotheses or helper lemma applications. The remaining long chains of calculations are performed automatically via SMT-based PLE. (That is, they must provide case statements and recursive structure, but *not* chains of `=.` applications.) To wit, with complete proof search, the above proofs shrink to:

```

fib3_2 :: {fib 3 = 2}   fibUp  :: Up fib   fMono :: Mono / [y]
fib3_2 = ()             fibUp  0 = ()     fMono f up x y

```

<pre> app_assoc :: AppendAssoc app_assoc [] ys zs = ([] ++ ys) ++ zs =. ys ++ zs =. [] ++ (ys ++ zs) ** QED app_assoc (x:xs) ys zs = ((x : xs) ++ ys) ++ zs =. (x : (xs ++ ys)) ++ zs =. x : ((xs ++ ys) ++ zs) ∴ app_assoc xs ys zs =. x : (xs ++ (ys ++ zs)) =. (x : xs) ++ (ys ++ zs) ** QED </pre>	<pre> app_assoc :: AppendAssoc app_assoc [] ys zs = () app_assoc (x:xs) ys zs = app_assoc xs ys zs </pre>
<pre> app_right_id :: AppendNilId app_right_id [] = () app_right_id (x:xs) = app_right_id xs </pre>	<pre> map_fusion :: MapFusion map_fusion f g [] = () map_fusion f g (x:xs) = map_fusion f g xs </pre>

Fig. 1. (L) Equational proof of append associativity, (R) PLE proof, also of append-id and map-fusion.

```

fibUp 1 = ()           | x+1 == y = up x
fibUp n = ()          | x+1 < y = up (y-1) &&& fMono up x (y-1)

```

where the combinator $p \ \&\& \ q = ()$ inserts the propositions p and q to the VC hypothesis.

PLE vs. Axiomatization Existing SMT based verifiers like DAFNY [Leino 2010] and F* [Swamy et al. 2016] use the classical *axiomatic* approach to verify assertions over user-defined functions like `fib`. In these systems, the function is encoded in the logic as a universally quantified formula (or axiom): $\forall n. \text{fib}P \ n$ after which the SMT solver may instantiate the above axiom at 3, 2, 1 and 0 in order to automatically prove $\{\text{fib } 3 = 2\}$.

The automation offered by axioms is a bit of a devil’s bargain, as axioms render VC checking *undecidable*, and in practice automatic axiom instantiation can easily lead to infinite “matching loops”. For example, the existence of a term `fib n` in a VC can trigger the above axiom, which may then produce the terms `fib (n - 1)` and `fib (n - 2)`, which may then recursively give rise to further instantiations *ad infinitum*. To prevent matching loops an expert must carefully craft “triggers” or, alternatively provide a “fuel” parameter [Amin et al. 2014] that bounds the depth of instantiation. Both these approaches ensure termination, but can cause the axiom to not be instantiated at the right places, thereby rendering the VC checking *incomplete*. The incompleteness is illustrated by the following example from the DAFNY benchmark suite [Leino 2016]

```

pos n | n < 0      = 0           test  :: y:{y > 5} → {pos n = 3 + pos (n-3)}
      | otherwise = 1 + pos (n-1) test _ = ()

```

DAFNY (and F*’s) fuel-based approach fails to check the above, when the fuel value is less than 3. One could simply raise-the-fuel-and-try-again but at what point does the user know when to stop? In contrast, PLE (1) does not require any fuel parameter, (2) is able to automatically perform the required unfolding to verify this example, *and* (3) is guaranteed to terminate.

2.5 Case Study: Laws for Lists

Reflection and PLE are not limited to integers. We end the overview by showing how they verify textbook properties of lists equipped with `append (++)` and `map` functions:

```

reflect (++) :: [a] → [a] → [a]   reflect map :: (a → b) → [a] → [b]
[]      ++ ys = ys                 map f []      = []
(x:xs) ++ ys = x : (xs ++ ys)     map f (x:xs) = f x : map f xs

```

In § 5.1 we will describe how the reflection mechanism illustrated via `fibP` is extended to account for ADTs using SMT-decidable selection and projection operations, which reflect the definition of `++` into the refinement as: `if isNil xs then ys else sel1 xs : (sel2 xs ++ ys)`. Note that `Liquid Haskell` automatically checks that `++` and `map` are total [Vazou et al. 2014a], which lets us safely **reflect** them into the refinement logic.

Laws We can specify various laws about lists with refinement types. For example, the below laws state that (1) appending to the right is an *identity* operation, (2) appending is an *associative* operation, and (3) `map` *distributes* over function composition:

```

type AppendNilId = xs:_ → { xs ++ [] = xs }
type AppendAssoc = xs:_ → ys:_ → zs:_ → { xs ++ (ys ++ zs) = (xs ++ ys) ++ zs }
type MapFusion   = f:_ → g:_ → xs:_ → { map (f . g) xs = map (f . map g) xs }

```

Proofs On the right in Figure 1 we show the proofs of these laws using PLE, which should be compared to the classical equational proof *e.g.* [Wadler 1987] shown on the left. With PLE, the user need only provide the high-level structure — the case splits and invocations of the induction hypotheses — after which PLE automatically completes the rest of the equational proof. Thus using SMT-based PLE, `app_assoc` shrinks down to its essence: an induction over the list `xs`. The difference is even more stark with `map_fusion` whose full equational proof is omitted, as it is twice as long.

PLE vs. Normalization The proofs in Figure 1 may remind readers familiar with Type-Theory based proof assistants (*e.g.* `Coq` or `Agda`) of the notions of *type-level normalization* and *rewriting* that permit similar proofs in those systems. While our approach of PLE is inspired by the idea of type level computation, it differs from it in two significant ways. First, from a *theoretical* point of view, SMT logics are not equipped with any notion of computation, normalization, canonicity or rewriting. Instead, our PLE algorithm shows how to *emulate* those ideas by asserting equalities corresponding to function definitions (Theorem 6.10). Second, from a *practical* perspective, the combination of (decidable) SMT-based theory reasoning and PLE’s proof search can greatly simplify verification. For example, consider the `swap` function from a `Coq` textbook [Appel 2016]:

```

swap :: [Integer] → [Integer]
swap (x1:x2:xs) = if x1 > x2 then x2:x1:x2 else x1:x2:xs
swap xs         = xs

```

In Figure 2 we show four proofs that `swap` is idempotent: Appel’s proof using `Coq` (simplified by the use of a hint database and the arithmetic tactic `omega`), its variant in `Agda` (for any Decidable Partial Order), the PLE proof, and a proof using the `DAFNY` verifier. It is readily apparent that PLE’s proof search working hand-in-glove with SMT-based theory reasoning makes proving the result relatively trivial. Of course, proof assistants like `Agda`, `Coq`, and `Isabelle` emit easily checkable certificates and have decades-worth of tactics, libraries and proof scripts that enable large scale proof engineering. On the other hand, `DAFNY`’s fuel-based axiom instantiation automatically unfolds the definition of `swap` twice, which thereby completing the proof without any user input. Note that these heuristics are orthogonal to PLE and can be combined with it, if the user wishes to trade off predictability for even more automation.

Summary We saw an overview of an SMT-automated refinement type checker that achieves SMT-decidable checking by restricting verification conditions to be *quantifier-free* and hence, decidable. In existing SMT-based verifiers (*e.g.* `DAFNY`) there are two main reasons to introduce quantifiers, namely (1) to express quantified *specifications*, and (2) to encode the semantics of *user-defined functions*. Next, we use propositions-as-types to encode quantified specifications and in § 4 we show how to encode the semantics of user-defined functions via refinement reflection.

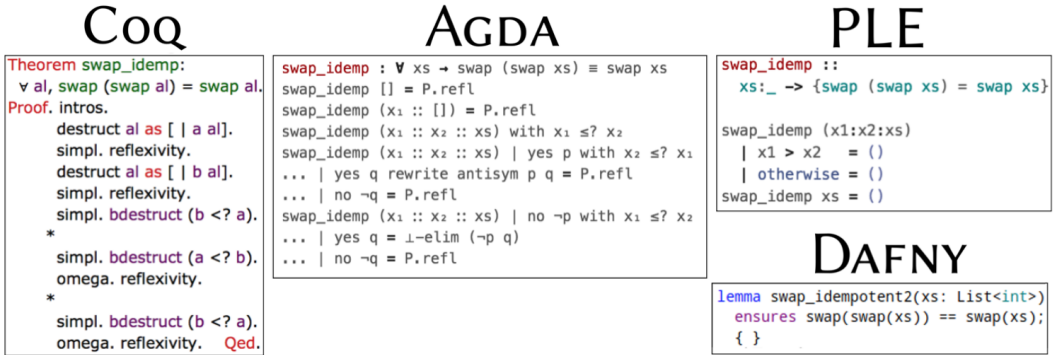


Fig. 2. Proofs that swap is idempotent with Coq, AGDA, DAFNY and PLE.

3 EMBEDDING NATURAL DEDUCTION WITH REFINEMENT TYPES

In this section we show how user-provided *quantified* specifications can be naturally encoded using λ -abstractions and dependent pairs to encode universal and existential quantification respectively. Proof terms can be generated using the standard natural deduction derivation rules, following Propositions as Types [Wadler 2015] (also known as the Curry-Howard isomorphism [Howard 1980]). What is new is that we exploit this encoding to show for the first time that a refinement type system can represent any proof in Gentzen’s natural deduction [Gentzen 1935] while still taking advantage of SMT decision procedures to automate the quantifier-free portion of natural deduction proofs. For simplicity, in this section we assume all terms are total; we formalize and relax this requirement in the sequel.

3.1 Propositions: Refinement Types

Figure 3 maps logical predicates to types constructed over quantifier-free refinements.

Native terms Native terms consist of all of the (quantifier-free) expressions of the refinement languages. In § 4 we formalize refinement typing in a core calculus λ^R where refinements include (quantifier-free) terminating expressions.

Boolean connectives Implication $\phi_1 \Rightarrow \phi_2$ is encoded as a *function* from the proof of ϕ_1 to the proof of ϕ_2 . Negation is encoded as an implication where the consequent is `False`. Conjunction $\phi_1 \wedge \phi_2$ is encoded as the *pair* (ϕ_1, ϕ_2) that contains the proofs of *both* conjuncts and disjunction $\phi_1 \vee \phi_2$ is encoded as the *sum* type `Either` that contains the proofs of *one of* the disjuncts, *i.e.* where `data Either a b = Left a | Right b`.

Quantifiers Universal quantification $\forall x.\phi$ is encoded as lambda abstraction $x : \tau \rightarrow \phi$ and eliminated by function application. Existential quantification $\exists x.\phi$ is encoded as a dependent pair $(x :: \tau, \phi)$ that contains the term x and a proof of a formula that depends on x . Even though refinement type systems do not traditionally come with explicit syntax for dependent pairs, one can encode dependent pairs in refinements using abstract refinement types [Vazou et al. 2013] which do not add extra complexity to the system. Consequently, we add the syntax for dependent pairs in Figure 3 as syntactic sugar for abstract refinements.

	Logical Formula	Refinement Type
Native Terms	e	$\{e\}$
Implication	$\phi_1 \Rightarrow \phi_2$	$\phi_1 \rightarrow \phi_2$
Negation	$\neg\phi$	$\phi \rightarrow \{\text{False}\}$
Conjunction	$\phi_1 \wedge \phi_2$	(ϕ_1, ϕ_2)
Disjunction	$\phi_1 \vee \phi_2$	Either $\phi_1 \phi_2$
Forall	$\forall x. \phi$	$x : \tau \rightarrow \phi$
Exists	$\exists x. \phi$	$(x :: \tau, \phi)$

Fig. 3. Mapping from logical predicates to quantifier-free refinement types. $\{e\}$ abbreviates $\{v : \text{Prop} \mid e\}$. Function binders are not relevant for negation and implication, and hence, elided.

3.2 Proofs: Natural Deduction

We overload ϕ to be both a proposition and a refinement type. We connect these two meanings of ϕ by using the Propositions as Types [Wadler 2015], to prove that if there exists an expression (or proof term) with refinement type ϕ , then the proposition ϕ is valid.

We construct proofs terms using Gentzen’s natural deduction system [Gentzen 1935], whose rules map directly to refinement type derivations. The rules for natural deduction arise from the propositions-as-types reading of the standard refinement type checking rule (to be defined in § 4) $\Gamma \vdash e : \phi$ as “ ϕ is provable under the assumptions of Γ ”. We write $\Gamma \vdash_{ND} \phi$ for Gentzen’s natural deduction judgement “under assumption Γ , proposition ϕ holds”. Then, each of Gentzen’s logical rules can be recovered from the rules in Figure 5 by rewriting each judgement $\Gamma \vdash e : \phi$ of λ^R as $\Gamma \vdash_{ND} \phi$. For example, conjunction and universal elimination can be derived as:

$$\frac{\Gamma \vdash_{ND} \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash_{ND} \phi \quad \Gamma, \phi_2 \vdash_{ND} \phi}{\Gamma \vdash_{ND} \phi} \vee\text{-E} \quad \frac{\Gamma \vdash_{ND} e_x \text{ term} \quad \Gamma \vdash_{ND} \forall x. \phi}{\Gamma \vdash_{ND} \phi[x/e_x]} \forall\text{-E}$$

Programs as Proofs As Figure 5 directly maps natural deduction rules to derivations that are accepted by refinement typing, we conclude that if there exists a natural deduction derivation for a proposition ϕ , then there exists an expression that has the refinement type ϕ .

THEOREM 3.1. *If $\Gamma \vdash_{ND} \phi$, then we can construct an e such that $\Gamma \vdash e : \phi$.*

Note that our embedding is *not* an isomorphism, since the inverse of Theorem 3.1 does not hold. As a counterexample, the law of the excluded middle (i.e. $p : \{\text{True}\} \vdash () : p \wedge \neg p$) is evident in our system, but cannot be proved using natural deduction (i.e. $\{\text{True}\} \not\vdash_{ND} p \wedge \neg p$). The reason for that is that our system is using the classical logic of the SMTs, that includes the law of the excluded middle. On the contrary, in intuitionistic systems that also encode natural deduction (e.g. Coq, IDRIS, NuPRL) the law of the excluded middles should be axiomatized.

3.3 Examples

Next, we illustrate our encoding with examples of proofs for quantified propositions ranging from textbook logical tautologies, properties of datatypes like lists, and induction on natural numbers.

Natural Deduction as Type Derivation We illustrate the mapping from natural deduction rules to typing rules in Figure 4 which uses typing judgments to express Gentzen’s proof of the proposition

$$\phi \equiv (\exists x. \forall y. (p \ x \ y)) \Rightarrow (\forall y. \exists x. (p \ x \ y))$$

Read bottom-up, the derivation provides a proof of ϕ . Read top-down, it constructs a *proof* of the formula as the *term* $\lambda e \ y. \text{case } e \text{ of } \{(x, e_x) \rightarrow (x, e_x \ y)\}$. This proof term corresponds directly to the following Haskell expression that typechecks with type ϕ .

$$\begin{array}{c}
\frac{e:\phi_e, y:\tau_y \vdash e:\phi_e \quad \frac{e:\phi_e, y:\tau_y, x:t_x, e_x:\phi_x \vdash e_x:\phi_x \quad e:\phi_e, y:\tau_y, x:t_x, e_x:\phi_x \vdash y:\tau_y}{e:\phi_e, y:\tau_y, x:t_x, e_x:\phi_x \vdash e_x y:p x y} \forall E}{e:\phi_e, y:\tau_y \vdash \text{case } e \text{ of } \{(x, e_x) \rightarrow (x, e_x y)\} : \exists x.(p x y)} \exists E \\
\frac{e:\phi_e \vdash \lambda y. \text{case } e \text{ of } \{(x, e_x) \rightarrow (x, e_x y)\} : \forall y. \exists x.(p x y)}{\emptyset \vdash \lambda e y. \text{case } e \text{ of } \{(x, e_x) \rightarrow (x, e_x y)\} : (\exists x. \forall y.(p x y)) \Rightarrow (\forall y. \exists x.(p x y))} \forall I \Rightarrow -I
\end{array}$$

Fig. 4. Proof of $(\exists x. \forall y.(p x y)) \Rightarrow (\forall y. \exists x.(p x y))$ where $\phi_e \equiv \exists x. \forall y.(p x y)$, $\phi_x \equiv \forall y.(p x y)$.

```

exAll :: p:(a -> a -> Bool) -> (x::a, y:a -> {p x y}) -> y:a -> (x::a, {p x y})
exAll e = \e y -> case e of {(x, ex) -> (x, ex y)}

```

SMT-aided proofs The great benefit of using refinement types to encode natural deduction is that the quantifier-free portions of the proof can be automated via SMTs. For every quantifier-free proposition ϕ , you can convert between $\{\phi\}$, where ϕ is treated as an SMT-proposition and ϕ , where ϕ is treated as a type; and this conversion goes *both* ways. For example, let $\phi \equiv p \wedge (q||r)$. Then `flatten` converts from ϕ to $\{\phi\}$ and `expand` the other way, while this conversion is SMT-aided.

```

flatten :: p:_ -> q:_ -> r:_ -> ({p}, Either {q} {r}) -> {p && (q || r)}
flatten (pf, Left qf) = pf &&& qf
flatten (pf, Right rf) = pf &&& rf

```

```

expand :: p:_ -> q:_ -> r:_ -> {p && (q || r)} -> ({p}, Either {q} {r})
expand proof | q = (proof, Left proof)
expand proof | r = (proof, Right proof)

```

Distributing Quantifiers Next, we construct the proof terms needed to prove two logical properties: that existentials distribute over disjunctions and forall over conjunctions, *i.e.*

$$\phi_{\exists} \equiv (\exists x. p x \vee q x) \Rightarrow ((\exists x. p x) \vee (\exists x. q x)) \quad (1)$$

$$\phi_{\forall} \equiv (\forall x. p x \wedge q x) \Rightarrow ((\forall x. p x) \wedge (\forall x. q x)) \quad (2)$$

The specification of these properties requires nesting quantifiers inside connectives and vice versa. The proof of ϕ_{\exists} (1) proceeds by existential case splitting and introduction:

```

exDistOr :: p:_ -> q:_ -> (x::a, Either {p x} {q x})
           -> Either (x::a, {p x}) (x::a, {q x})
exDistOr _ _ (x, Left px) = Left (x, px)
exDistOr _ _ (x, Right qx) = Right (x, qx)

```

Dually, we prove ϕ_{\forall} (2) via a λ -abstraction and case spitting inside the conjunction pair:

```

allDistAnd :: p:_ -> q:_ -> (x:a -> ({p x}, {q x}))
            -> ((x:a -> {p x}), (x:a -> {q x}))
allDistAnd _ _ andx = ( (\x -> case andx x of (px, _) -> px
                        , (\x -> case andx x of (_, qx) -> qx) )

```

The above proof term exactly corresponds to its natural deduction proof derivation but using SMT-aided verification can get simplified to the following

```

allDistAnd _ _ andx = (pf, pf)
where pf x = case andx x of (px, py) -> px &&& py

```

$$\begin{array}{c}
\frac{\Gamma \vdash \text{fst } e : \phi_1 \quad \Gamma \vdash \text{snd } e : \phi_2}{\Gamma \vdash e : (\phi_1, \phi_2)} \wedge\text{-I} \\
\frac{\Gamma \vdash e_1 : \phi_1}{\Gamma \vdash \text{Left } e_1 : \text{Either } \phi_1 \phi_2} \vee\text{-L-I} \\
\frac{\Gamma \vdash e_2 : \phi_2}{\Gamma \vdash \text{Right } e_2 : \text{Either } \phi_1 \phi_2} \vee\text{-R-I} \\
\frac{\Gamma, x : \phi_x \vdash e : \phi}{\Gamma \vdash \lambda x. e : \phi_x \rightarrow \phi} \Rightarrow\text{-I} \\
\frac{\Gamma, x : \tau \vdash e : \phi}{\Gamma \vdash \lambda x. e : (x : \tau \rightarrow \phi)} \forall\text{-I} \\
\frac{\Gamma \vdash \text{fst } e : \tau \quad \Gamma, x : \tau \vdash \text{snd } e : \phi}{\Gamma \vdash e : (x :: \tau, \phi[x/\text{fst } e])} \exists\text{-I} \\
\frac{\Gamma \vdash e : (\phi_1, \phi_2)}{\Gamma \vdash \text{fst } e : \phi_1} \wedge\text{-L-E} \\
\frac{\Gamma \vdash e : (\phi_1, \phi_2)}{\Gamma \vdash \text{snd } e : \phi_2} \wedge\text{-R-E} \\
\frac{\Gamma \vdash e : \text{Either } \phi_1 \phi_2 \quad \Gamma, x_1 : \phi_1 \vdash e_1 : \phi \quad \Gamma, x_2 : \phi_2 \vdash e_2 : \phi}{\Gamma \vdash \text{case } e \text{ of } \{\text{Left } x_1 \rightarrow e_1; \text{Right } x_2 \rightarrow e_2\} : \phi} \vee\text{-E} \\
\frac{\Gamma \vdash e : \phi_x \rightarrow \phi \quad \Gamma \vdash e_x : \phi_x}{\Gamma \vdash e e_x : \phi} \Rightarrow\text{-E} \\
\frac{\Gamma \vdash e_x : \tau \quad \Gamma \vdash e : (x : \tau \rightarrow \phi)}{\Gamma \vdash e e_x : \phi[x/e_x]} \forall\text{-E} \\
\frac{\Gamma \vdash e : (x :: \tau, \phi_x) \quad \Gamma, x : \tau, y : \phi_x \vdash e' : \phi}{\Gamma \vdash \text{case } e \text{ of } \{(x, y) \rightarrow e'\} : \phi} \exists\text{-E}
\end{array}$$

Fig. 5. Natural deduction rules for refinement types. With $[\text{fst}|\text{snd}] e \equiv \text{case } e \text{ of } \{(x_1, x_2) \rightarrow [x_1|x_2]\}$.

Properties of User Defined Datatypes As ϕ can describe properties of data types like lists, we can prove properties of such types, e.g. that for every list xs , if there exists a list ys such that $xs == ys ++ ys$, then xs has even length.

$$\phi \equiv \forall xs. ((\exists ys. xs = ys ++ ys) \Rightarrow (\exists n. \text{len } xs = n + n))$$

The proof (`evenLen`) proceeds by existential elimination and introduction, and uses the `lenAppend` lemma, which uses induction on the input list and PLE to automate equational reasoning.

```

evenLen :: xs:[a] -> (ys:[a], {xs = ys ++ ys}) -> (n::Int, {len xs = n+n})
evenLen xs (ys, pf) = (len ys, lenAppend ys ys &&& pf)

lenAppend :: xs:_ -> ys:_ -> {len (xs ++ ys) = len xs + len ys}
lenAppend [] _ = ()
lenAppend (x:xs) ys = lenAppend xs ys

```

Induction on Natural Numbers Finally, we specify and verify *induction* on natural numbers:

$$\phi_{\text{ind}} \equiv (p \ 0 \wedge (\forall n. p \ (n - 1) \Rightarrow p \ n) \Rightarrow \forall n. p \ n)$$

The proof proceeds by induction (e.g. case splitting). In the base case, $n == 0$, the proof calls the left conjunct, which contains a proof of the base case. Otherwise, $0 < n$, the proof *applies* the induction hypothesis to the right conjunct instantiated at $n-1$.

```

ind :: p:_ -> ({p 0}, (n:Nat -> {p (n-1)} -> {p n})) -> n:Nat -> {p n}
ind p (p0, pn) 0 = p0
ind p (p0, pn) n = pn n (ind p (p0, pn) (n-1))

```

3.4 Consequences

To summarize, we use the propositions-as-types principle to make two important contributions. First, we show that natural deduction reasoning can smoothly co-exist with SMT-based verification to automate the decidable, quantifier-free portions of the proof.

Ops. \odot ::= = <	Preds. p ::= $p \bowtie p$ $\oplus_1 p$
Consts. c ::= \wedge ! \odot +, -, ...	n b x D $x \bar{p}$
True False 0, 1, ...	if p then p else p
Vals. w ::= c $\lambda x. e$ $D \bar{w}$	Ints. n ::= 0, -1, 1, ...
Exprs. e ::= w x $e e$	Bools. b ::= True False
case $x = e$ of $\{D \bar{x} \rightarrow e\}$	Bin-Ops. \bowtie ::= = < \wedge +, -, ...
Binds. b ::= e let rec $x : \tau = b$ in b	Un-Ops. \oplus_1 ::= ! ...
Progs. p ::= b reflect $x : \tau = e$ in p	Args. s_a ::= Int Bool U
Bas. Types B ::= Int Bool T	Fun $s_a s_a$
Ref. Types τ ::= $\{v : B^{[U]} \mid e\}$ $x : \tau_x \rightarrow \tau$	Sorts s ::= $s_a \mid s_a \rightarrow s$

Fig. 6. (Left) Syntax of λ^R : Denotational Typing. (Right) Syntax of λ^S : Algorithmic Typing.

Second, we show for first time how natural deduction proofs can be encoded in refinement type systems like LIQUID HASSELL and we expect this encoding to extend, in a straight-forward manner to other SMT-based deductive verifiers (e.g. DAFNY and F*). This encoding shows that refinement type systems are expressive enough to encode any intuitionistic natural deduction proof, gives a guideline for encoding proofs with nested quantifiers, and provides a pleasant implementation of natural deduction that is pedagogically useful.

4 REFINEMENT REFLECTION: λ^R

Refinement reflection encodes recursive functions in the quantifier-free, SMT logic and it is formalized in three steps. First, we develop a core calculus λ^R with an *undecidable* type system based on denotational semantics. We show how the soundness of the type system allows us to *prove theorems* using λ^R . Next, in § 5 we define a language λ^S that soundly approximates λ^R while enabling decidable SMT-based type checking. Finally, in § 6 we develop a complete proof search algorithm to automate equational reasoning.

4.1 Syntax

Figure 6 summarizes the syntax of λ^R , which is essentially the calculus λ^U [Vazou et al. 2014a] with explicit recursion and a special reflect binding to denote terms that are reflected into the refinement logic. The elements of λ^R are constants, values, expressions, binders and programs.

Constants The constants of λ^R include primitive relations \odot , here, the set $\{=, <\}$. Moreover, they include the booleans True, False, integers $-1, 0, 1$, etc., and logical operators $\wedge, !$, etc.

Data Constructors Data constructors are special constants. For example, the data type $[Int]$, which represents finite lists of integers, has two data constructors: $[]$ (nil) and $:$ (cons).

Values & Expressions The values of λ^R include constants, λ -abstractions $\lambda x. e$, and fully applied data constructors D that wrap values. The expressions of λ^R include values, variables x , applications $e e$, and case expressions.

Binders & Programs A *binder* b is a series of possibly recursive let definitions, followed by an expression. A *program* p is a series of reflect definitions, each of which names a function that is reflected into the refinement logic, followed by a binder. The stratification of programs via binders is required so that arbitrary recursive definitions are allowed in the program but cannot be inserted

into the logic via refinements or reflection. (We *can* allow non-recursive let binders in expressions e , but omit them for simplicity.)

4.2 Operational Semantics

We define \hookrightarrow to be the small step, call-by-name β -reduction semantics for λ^R . We evaluate reflected terms as recursive let bindings, with termination constraints imposed by the type system:

$$\text{reflect } x : \tau = e \text{ in } p \hookrightarrow \text{let rec } x : \tau = e \text{ in } p$$

We define \hookrightarrow^* to be the reflexive, transitive closure of \hookrightarrow . Moreover, we define \approx_β to be the reflexive, symmetric, and transitive closure of \hookrightarrow .

Constants Application of a constant requires the argument be reduced to a value; in a single step, the expression is reduced to the output of the primitive constant operation, i.e. $c \ v \hookrightarrow \delta(c, v)$. For example, consider $=$, the primitive equality operator on integers. We have $\delta(=, n) \doteq =_n$ where $\delta(=, m)$ equals True iff m is the same as n .

Equality We assume that the equality operator is defined for *all* values, and, for functions, is defined as extensional equality. That is, for all f and f' , $(f = f') \hookrightarrow \text{True}$ iff $\forall v. f \ v \approx_\beta f' \ v$. We assume source *terms* only contain implementable equalities over non-function types; while function extensional equality only appears in *refinements*.

4.3 Types

λ^R types include basic types, which are *refined* with predicates, and dependent function types. *Basic types* B comprise integers, booleans, and a family of data-types T (representing lists, trees *etc.*). For example, the data type $[Int]$ represents lists of integers. We refine basic types with predicates (boolean-valued expressions e) to obtain *basic refinement types* $\{v : B \mid e\}$. We use \Downarrow to mark provably terminating computations and use refinements to ensure that if $e : \{v : B \mid e'\}$, then e terminates. As discussed by Vazou et al. [2014a] termination labels can be checked using refinement types and are used to ensure that refinements cannot diverge as required for soundness of type checking under lazy evaluation. Termination checking is crucial for this work, as combined with syntactic checks for exhaustive definitions, it ensures totality (well-formedness) of expressions as required both by propositions-as-types (§ 3) and termination of PLE (§ 6). Finally, we have dependent *function types* $x : \tau_x \rightarrow \tau$ where the input x has the type τ_x and the output τ may refer to the input binder x . We write B to abbreviate $\{v : B \mid \text{True}\}$, and $\tau_x \rightarrow \tau$ to abbreviate $x : \tau_x \rightarrow \tau$ if x does not appear in τ .

Denotations Each type τ *denotes* a set of expressions $\llbracket \tau \rrbracket$, that is defined via the operational semantics [Knowles and Flanagan 2010]. Let $\text{shape}(\tau)$ be the type we get if we erase all refinements from τ and $e : \text{shape}(\tau)$ be the standard typing relation for the typed lambda calculus. Then, we define the denotation of types as:

$$\begin{aligned} \llbracket \{x : B \mid r\} \rrbracket &\doteq \{e \mid e : B, \text{ if } e \hookrightarrow^* w \text{ then } r[x/w] \hookrightarrow^* \text{True}\} \\ \llbracket \{x : B \Downarrow \mid r\} \rrbracket &\doteq \llbracket \{x : B \mid r\} \rrbracket \cap \{e \mid \exists w. e \hookrightarrow^* w\} \\ \llbracket x : \tau_x \rightarrow \tau \rrbracket &\doteq \{e \mid e : \text{shape}(\tau_x \rightarrow \tau), \forall e_x \in \llbracket \tau_x \rrbracket. (e \ e_x) \in \llbracket \tau[x/e_x] \rrbracket\} \end{aligned}$$

Constants For each constant c we define its type $\text{prim}(c)$ such that $c \in \llbracket \text{prim}(c) \rrbracket$. For example,

$$\begin{aligned} \text{prim}(3) &\doteq \{v : \text{Int} \Downarrow \mid v = 3\} \\ \text{prim}(+) &\doteq x : \text{Int} \Downarrow \rightarrow y : \text{Int} \Downarrow \rightarrow \{v : \text{Int} \Downarrow \mid v = x + y\} \\ \text{prim}(\leq) &\doteq x : \text{Int} \Downarrow \rightarrow y : \text{Int} \Downarrow \rightarrow \{v : \text{Bool} \Downarrow \mid v \Leftrightarrow x \leq y\} \end{aligned}$$

4.4 Refinement Reflection

Reflection *strengthens* function output types with a refinement that *reflects* the definition of the function in the logic. We do this by treating each `reflect`-binder (`reflect f : $\tau = e$ in p`) as a `let rec`-binder (`let rec f : $\text{Reflect}(\tau, e) = e$ in p`) during type checking (rule T-REFL in Figure 7).

Reflection We write $\text{Reflect}(\tau, e)$ for the *reflection* of the term e into the type τ , defined as

$$\begin{aligned} \text{Reflect}(\{v : B^\Downarrow \mid r\}, e) &\doteq \{v : B^\Downarrow \mid r \wedge v = e\} \\ \text{Reflect}(x : \tau_x \rightarrow \tau, \lambda x. e) &\doteq x : \tau_x \rightarrow \text{Reflect}(\tau, e) \end{aligned}$$

As an example, recall from § 2 that the **reflect** `fib` strengthens the type of `fib` with the refinement `fibP`. That is, let the user specified type of `fib` be t_{fib} and the its definition be definition $\lambda n. e_{\text{fib}}$.

$$\begin{aligned} t_{\text{fib}} &\doteq \{v : \text{Int}^\Downarrow \mid 0 \leq v\} \rightarrow \{v : \text{Int}^\Downarrow \mid 0 \leq v\} \\ e_{\text{fib}} &\doteq \text{case } x = n \leq 1 \text{ of } \{\text{True} \rightarrow n; \text{False} \rightarrow \text{fib}(n-1) + \text{fib}(n-2)\} \end{aligned}$$

Then, the reflected type of `fib` will be:

$$\text{Reflect}(t_{\text{fib}}, e_{\text{fib}}) = n : \{v : \text{Int}^\Downarrow \mid 0 \leq v\} \rightarrow \{v : \text{Int}^\Downarrow \mid 0 \leq v \wedge v = e_{\text{fib}}\}$$

Termination Checking We defined $\text{Reflect}(\cdot, \cdot)$ to be a *partial* function that only reflects provably terminating expressions, *i.e.* expressions whose result type is marked with \Downarrow . If a non-provably terminating function is reflected in an λ^R expression then type checking will fail (with a reflection type error in the implementation). This restriction is crucial for soundness, as diverging expressions can lead to inconsistencies. For example, reflecting the diverging $f \ x = 1 + f \ x$ into the logic leads to an inconsistent system that is able to prove $0 = 1$.

Automatic Reflection Reflection of λ^R expressions into the refinements happens automatically by the type system, not manually by the user. The user simply annotates a function f as `reflect f`. Then, the rule T-REFL in Figure 7 is used to type check the reflected function by strengthening the f 's result via $\text{Reflect}(\cdot, \cdot)$. Finally, the rule T-LET is used to check that the automatically strengthened type of f satisfies f 's implementation.

4.5 Typing Rules

Next, we present the type-checking rules of λ^R , as found in Figure 7.

Environments and Closing Substitutions A *type environment* Γ is a sequence of type bindings $x_1 : \tau_1, \dots, x_n : \tau_n$. An environment denotes a set of *closing substitutions* θ which are sequences of expression bindings: $x_1 \mapsto e_1, \dots, x_n \mapsto e_n$ such that:

$$[[\Gamma]] \doteq \{\theta \mid \forall x : \tau \in \Gamma. \theta(x) \in [[\theta \cdot \tau]]\}$$

where $\theta \cdot \tau$ applies a substitution to a type (and likewise $\theta \cdot p$, to a program).

A reflection environment R is a sequence that binds the names of the reflected functions with their definitions $f_1 \mapsto e_1, \dots, f_n \mapsto e_n$. A reflection environment respects a type environment when all reflected functions satisfy their types:

$$\Gamma \models R \doteq \forall (f \mapsto e) \in R. \exists \tau. (f : \tau) \in \Gamma \wedge (\Gamma; R \vdash e : \tau)$$

Typing A judgment $\Gamma; R \vdash p : \tau$ states that the program p has the type τ in the type environment Γ and the reflection environment R . That is, when the free variables in p are bound to expressions described by Γ , the program p will evaluate to a value described by τ .

Rules All but two of the rules are the standard refinement typing rules [Knowles and Flanagan 2010; Vazou et al. 2014a] except for the addition of the reflection environment R at each rule. First, rule T-REFL is used to extend the reflection environment with the binding of the function name

Typing $\Gamma; R \vdash p : \tau$

$$\begin{array}{c}
\frac{x : \tau \in \Gamma}{\Gamma; R \vdash x : \tau} \text{T-VAR} \quad \frac{}{\Gamma; R \vdash c : \text{prim}(c)} \text{T-CON} \quad \frac{\Gamma; R \vdash p : \tau' \quad \Gamma; R \vdash \tau' \leq \tau}{\Gamma; R \vdash p : \tau} \text{T-SUB} \\
\frac{\Gamma; R \vdash e : \{v : B \mid e_r\}}{\Gamma; R \vdash e : \{v : B \mid e_r \wedge v = e\}} \text{T-EXACT} \quad \frac{\Gamma, x : \tau_x; R \vdash e : \tau}{\Gamma; R \vdash \lambda x. e : (x : \tau_x \rightarrow \tau)} \text{T-FUN} \\
\frac{\Gamma; R \vdash e_1 : (x : \tau_x \rightarrow \tau) \quad \Gamma; R \vdash e_2 : \tau_x}{\Gamma; R \vdash e_1 e_2 : \tau} \text{T-APP} \quad \frac{\Gamma, x : \tau_x; R \vdash b_x : \tau_x \quad \Gamma, x : \tau_x \vdash \tau_x \quad \Gamma, x : \tau_x; R \vdash b : \tau \quad \Gamma \vdash \tau}{\Gamma; R \vdash \text{let rec } x : \tau_x = b_x \text{ in } b : \tau} \text{T-LET} \\
\frac{\Gamma; R \vdash e : \{v : T \mid e_r\} \quad \forall i. \text{prim}(D_i) = \overline{y_j} : \overline{\tau_j} \rightarrow \{v : T \mid e_{r_i}\} \quad \Gamma, \overline{y_j} : \overline{\tau_j}, x : \{v : T \mid e_r \wedge e_{r_i}\}; R \vdash e_i : \tau}{\Gamma; R \vdash \text{case } x = e \text{ of } \{D_i \overline{y_i} \rightarrow e_i\} : \tau} \text{T-CASE} \\
\frac{\Gamma; R, f \mapsto e \vdash \text{let rec } f : \text{Reflect}(\tau_f, e) = e \text{ in } p : \tau}{\Gamma; R \vdash \text{reflect } f : \tau_f = e \text{ in } p : \tau} \text{T-REFL}
\end{array}$$

Well Formedness $\Gamma \vdash \tau$

$$\frac{\Gamma, v : B; \emptyset \vdash e : \text{Bool}^\Downarrow}{\Gamma \vdash \{v : B \mid e\}} \text{WF-BASE} \quad \frac{\Gamma \vdash \tau_x \quad \Gamma, x : \tau_x \vdash \tau}{\Gamma \vdash x : \tau_x \rightarrow \tau} \text{WF-FUN}$$

Subtyping $\Gamma; R \vdash \tau_1 \leq \tau_2$

$$\frac{\forall \theta \in \llbracket \Gamma \rrbracket. \llbracket \theta \cdot \{v : B \mid e_1\} \rrbracket \subseteq \llbracket \theta \cdot \{v : B \mid e_2\} \rrbracket}{\Gamma; R \vdash \{v : B \mid e_1\} \leq \{v : B \mid e_2\}} \le\text{-BASE-}\lambda^R \\
\frac{\Gamma; R \vdash \tau'_x \leq \tau_x \quad \Gamma, x : \tau'_x; R \vdash \tau \leq \tau'}{\Gamma; R \vdash x : \tau_x \rightarrow \tau \leq x : \tau'_x \rightarrow \tau'} \le\text{-FUN}$$

Fig. 7. Typing of λ^R

with its definition ($f \mapsto e$) and moreover to strengthen the type of each reflected binder with its definition, as described previously in § 4.4. Second, rule T-EXACT strengthens the expression with a singleton type equating the value and the expression (*i.e.* reflecting the expression in the type). This is a generalization of the “selffication” rules from [Knowles and Flanagan 2010; Ou et al. 2004] and is required to equate the reflected functions with their definitions. For example, the application `fib 1` is typed as $\{v : \text{Int}^\Downarrow \mid \text{fibP } 1 \wedge v = \text{fib } 1\}$ where the first conjunct comes from the (reflection-strengthened) output refinement of `fib` § 2 and the second comes from rule T-EXACT.

Well-formedness A judgment $\Gamma \vdash \tau$ states that the refinement type τ is well-formed in the environment Γ . Following Vazou et al. [2014a], τ is well-formed if all the refinements in τ are Bool-typed, provably terminating expressions in Γ .

Subtyping A judgment $\Gamma; R \vdash \tau_1 \leq \tau_2$ states that the type τ_1 is a subtype of τ_2 in the environments Γ and R . Informally, τ_1 is a subtype of τ_2 if, when the free variables of τ_1 and τ_2 are bound to expressions described by Γ , the denotation of τ_1 is *contained in* the denotation of τ_2 . Subtyping of basic types reduces to denotational containment checking, shown in rule $\le\text{-BASE-}\lambda^R$. That is, τ_1 is a subtype of τ_2 under Γ if for any closing substitution θ in $\llbracket \Gamma \rrbracket$, $\llbracket \theta \cdot \tau_1 \rrbracket$ is contained in $\llbracket \theta \cdot \tau_2 \rrbracket$.

Soundness Following λ^U [Vazou et al. 2014a], in Vazou et al. [2017] we prove that evaluation preserves typing and typing implies denotational inclusion.

THEOREM 4.1. [Soundness of λ^R]

- **Denotations** If $\Gamma; R \vdash p : \tau$ then $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot p \in \llbracket \theta \cdot \tau \rrbracket$.
- **Preservation** If $\emptyset; \emptyset \vdash p : \tau$ and $p \hookrightarrow^* w$, then $\emptyset; \emptyset \vdash w : \tau$.

Theorem 4.1 lets us prove that if ϕ is a λ^R type interpreted as a proposition (using the mapping of Figure 3) and if there exists a p so that $\emptyset; \emptyset \vdash p : \phi$, the ϕ is valid. For example, in § 2 we verified that the term `fibUp` proves $n : \text{Nat} \rightarrow \{\text{fib } n \leq \text{fib } (n + 1)\}$. Via soundness of λ^R , we get that for each valid input n , the result refinement is valid.

$$\forall n. 0 \leq n \hookrightarrow^* \text{True} \Rightarrow \text{fib } n \leq \text{fib } (n + 1) \hookrightarrow^* \text{True}$$

5 ALGORITHMIC CHECKING: λ^S

λ^S is a first order approximation of λ^R where higher-order features are approximated with uninterpreted functions and the undecidable type subsumption rule $\leq\text{-BASE-}\lambda^R$ is replaced with a decidable one (i.e., $\leq\text{-BASE-PLE}$), yielding an sound and decidable SMT-based algorithmic type system. Figure 6 summarizes the syntax of λ^S , the *sorted* (SMT-) decidable logic of quantifier-free equality, uninterpreted functions and linear arithmetic (QF-EUFLIA) [Barrett et al. 2010; Nelson 1980]. The *terms* of λ^S include integers n , booleans b , variables x , data constructors D (encoded as constants), fully applied unary \oplus_1 and binary \bowtie operators, and application $x \bar{p}$ of an uninterpreted function x . The *sorts* of λ^S include built-in integer `Int` and `Bool` for representing integers and booleans. The interpreted functions of λ^S , e.g. the logical constants `=` and `<`, have the function sort $s \rightarrow s$. Other functional values in λ^R , e.g. reflected λ^R functions and λ -expressions, are represented as first-order values with the uninterpreted sort `Fun s s`. The sort `U` represents all other values.

5.1 Transforming λ^R into λ^S

The judgment $\Gamma \vdash e \rightsquigarrow p$ states that a λ^R term e is transformed, under an environment Γ , into a λ^S term p . If $\Gamma \vdash e \rightsquigarrow p$ and Γ is clear from the context we write $\llbracket e \rrbracket$ and $\llbracket p \rrbracket$ to denote the translation from λ^R to λ^S and back. Most of the transformation rules are identity and can be found in [Vazou et al. 2017]. Here we discuss the non-identity ones.

Embedding Types We embed λ^R types into λ^S sorts as:

$$\begin{array}{lll} \llbracket \text{Int} \rrbracket \doteq \text{Int} & \llbracket T \rrbracket \doteq \text{U} & \llbracket \{v : B^{\llbracket U \rrbracket} \mid e\} \rrbracket \doteq \llbracket B \rrbracket \\ \llbracket \text{Bool} \rrbracket \doteq \text{Bool} & & \llbracket x : \tau_x \rightarrow \tau \rrbracket \doteq \text{Fun } \llbracket \tau_x \rrbracket \llbracket \tau \rrbracket \end{array}$$

Embedding Constants Elements shared on both λ^R and λ^S translate to themselves. These elements include booleans, integers, variables, binary and unary operators. SMT solvers do not support currying, and so in λ^S , all function symbols must be fully applied. Thus, we assume that all applications to primitive constants and data constructors are fully applied, e.g. by converting source terms like `(+ 1)` to `(\lambda z \rightarrow z + 1)`.

Embedding Functions As λ^S is first-order, we embed λ s using the uninterpreted function `lam`.

$$\frac{\Gamma, x : \tau_x \vdash e \rightsquigarrow p \quad \Gamma; \emptyset \vdash (\lambda x. e) : (x : \tau_x \rightarrow \tau)}{\Gamma \vdash \lambda x. e \rightsquigarrow \text{lam}_{\llbracket \tau_x \rrbracket}^{\llbracket \tau \rrbracket}} x p$$

The term $\lambda x. e$ of type $\tau_x \rightarrow \tau$ is transformed to `lamsx` $x p$ of sort `Fun sx s`, where s_x and s are respectively $\llbracket \tau_x \rrbracket$ and $\llbracket \tau \rrbracket$, `lamsx` is a special uninterpreted function of sort $s_x \rightarrow s \rightarrow \text{Fun } s_x s$, and

x of sort s_x and r of sort s are the embedding of the binder and body, respectively. As `lam` is an SMT-function, it *does not* create a binding for x . Instead, x is renamed to a *fresh* SMT name.

Embedding Applications We embed applications via defunctionalization [Reynolds 1972] using the uninterpreted app:

$$\frac{\Gamma \vdash e' \rightsquigarrow p' \quad \Gamma \vdash e \rightsquigarrow p \quad \Gamma; \emptyset \vdash e : \tau_x \rightarrow \tau}{\Gamma \vdash e e' \rightsquigarrow \text{app}_{\tau}^{\tau_x} p p'}$$

The term $e e'$, where e and e' have types $\tau_x \rightarrow \tau$ and τ_x , is transformed to $\text{app}_{\tau}^{\tau_x} p p' : s$ where s and s_x are $[\tau]$ and $[\tau_x]$, the $\text{app}_{\tau}^{\tau_x}$ is a special uninterpreted function of sort $\text{Fun } s_x s \rightarrow s_x \rightarrow s$, and p and p' are the respective translations of e and e' .

Embedding Data Types We embed data constructors to a predefined λ^S constant s_D of sort $[\text{prim}(D)] : \Gamma \vdash D \rightsquigarrow s_D$. For each datatype, we create reflected measures that *check* the top-level constructor and *select* their individual fields. For example, for lists, we create measures

$$\begin{array}{lll} \text{isNil } [] & = \text{True} & \text{isCons } (x:xs) = \text{True} & \text{sel1 } (x:xs) = x \\ \text{isNil } (x:xs) & = \text{False} & \text{isCons } [] = \text{False} & \text{sel2 } (x:xs) = xs \end{array}$$

The above selectors can be modeled precisely in the refinement logic via SMT support for ADTs [Nelson 1980]. To generalize, let D_i be a data constructor such that $\text{prim}(D_i) \doteq \tau_{i,1} \rightarrow \dots \rightarrow \tau_{i,n} \rightarrow \tau$. Then *check* is_{D_i} has the sort $\text{Fun } [\tau] \text{Bool}$ and *select* $\text{sel}_{D_i,j}$ has the sort $\text{Fun } [\tau] [\tau_{i,j}]$.

Embedding Case Expressions We translate case-expressions of λ^R into nested `if` terms in λ^S , by using the check functions in the guards and the select functions for the binders of each case.

$$\frac{\Gamma \vdash e \rightsquigarrow p \quad \Gamma \vdash e_i[\overline{y_i/\text{sel}_{D_i} x}][x/e] \rightsquigarrow p_i}{\Gamma \vdash \text{case } x = e \text{ of } \{D_i \overline{y_i} \rightarrow e_i\} \rightsquigarrow \text{if app is}_{D_i} p \text{ then } p_1 \text{ else } \dots \text{ else } p_n}$$

The above translation yields the reflected definition for `append` (`++`) from (§ 2.5).

Semantic Preservation The translation preserves the semantics of the expressions. Informally, if $\Gamma \vdash e \rightsquigarrow p$, then for every substitution θ and every logical model σ that respects the environment Γ if $\theta \cdot e \hookrightarrow^* v$ then $\sigma \models p = [v]$.

5.2 Algorithmic Type Checking

We make the type checking from Figure 7 algorithmic by checking subtyping via our novel, SMT-based *Proof by Logical Evaluation* (PLE). Next, we formalize how PLE makes checking algorithmic and in § 6 we describe the PLE procedure in detail.

Verification Conditions Recall that in § 5.1 we defined $[\cdot]$ as the translation from λ^R to λ^S . Informally, the implication or *verification condition* (VC) $[\Gamma] \Rightarrow p_1 \Rightarrow p_2$ is *valid* only if the set of values described by p_1 is subsumed by the set of values described by p_2 under the assumptions of Γ . Γ is embedded into logic by conjoining the refinements of terminating binders [Vazou et al. 2014a]:

$$[\Gamma] \doteq \bigcup_{x \in \Gamma} [\Gamma, x] \quad \text{where we embed each binder as } \quad [\Gamma, x] \doteq \begin{cases} [e] & \text{if } \Gamma(x) = \{x : B^{\sharp} \mid e\} \\ \text{True} & \text{otherwise.} \end{cases}$$

Validity Checking Instead of directly using the VCs to check validity of programs, we use the procedure PLE that strengthens the assumption environment $[\Gamma]$ with equational properties. Concretely, given a reflection environment R , type environment Γ , and expression e , the procedure $\text{PLE}([\Gamma], [\Gamma], [e])$ – we will define $[R]$ in § 6.1 – returns *true* only when the expression e evaluates to `True` under the reflection and type environments R and Γ .

Terms	p, t, b	$::=$	λ^S if-free predicates from Figure 6
Functions	F	$::=$	$\lambda \bar{x}. \langle p \Rightarrow b \rangle$
Definitional Environment	Ψ	$::=$	$\emptyset \mid f \mapsto F, \Psi$
Logical Environment	Φ	$::=$	$\emptyset \mid p, \Phi$

Fig. 8. Syntax of Predicates, Terms and Reflected Functions

Subtyping via VC Validity Checking We make subtyping, and hence, typing decidable, by replacing the denotational base subtyping rule $\leq\text{-BASE-}\lambda^R$ with the conservative, algorithmic version $\leq\text{-BASE-PLE}$ that uses PLE to check the validity of the subtyping.

$$\frac{\text{PLE}(\llbracket R \rrbracket, \llbracket \Gamma, v : \{v : B^\Downarrow \mid e\} \rrbracket, \llbracket e' \rrbracket)}{\Gamma; R \vdash_{\text{PLE}} \{v : B \mid e\} \leq \{v : B \mid e'\}} \leq\text{-BASE-PLE}$$

This typing rule is sound as functions reflected in R always respect the typing environment Γ (by construction) and because PLE is sound (Theorem 6.2).

LEMMA 5.1. *If $\Gamma; R \vdash_{\text{PLE}} \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$ then $\Gamma; R \vdash \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$.*

Soundness of λ^S We write $\Gamma; R \vdash_{\text{PLE}} e : \tau$ for the judgments that can be derived by the algorithmic subtyping rule $\leq\text{-BASE-}\lambda^S$ (instead of $\leq\text{-BASE-}\lambda^R$.) Lemma 5.1 implies the soundness of λ^S .

THEOREM 5.2 (SOUNDNESS OF λ^S). *If $\Gamma; R \vdash_{\text{PLE}} e : \tau$ then $\Gamma; R \vdash e : \tau$.*

6 COMPLETE VERIFICATION: PROOF BY LOGICAL EVALUATION

Next, we formalize our Proof By Logical Evaluation algorithm PLE and show that it is sound (§ 6.1), that it is complete with respect to equational proofs (§ 6.2), and that it terminates (§ 6.3).

6.1 Algorithm

Figure 8 describes the input environments for PLE. The logical environment Φ contains a set of hypotheses p , described in Figure 6. The definitional environment Ψ maps function symbols f to their definitions $\lambda \bar{x}. \langle p \Rightarrow b \rangle$, written as λ -abstractions over guarded bodies. Moreover, the body b and the guard p contain neither λ nor if . These restrictions do not impact expressiveness: λ s can be named and reflected, and if -expressions can be pulled out into top-level guards using $\text{DeIf}(\cdot)$, found in Appendix [Vazou et al. 2017]. A definitional environment Ψ can be constructed from R as

$$\llbracket R \rrbracket \doteq \{f \mapsto \lambda \bar{x}. \text{DeIf}(\llbracket e \rrbracket) \mid (f \mapsto \lambda \bar{x}. e) \in R\}$$

Notation We write $f(\bar{t}) < \Phi$ if the λ^S term $(\text{app } \dots (\text{app } f \ t_1) \dots t_n)$ is a syntactic subterm of some $t' \in \Phi$. We abuse notation to write $f(\bar{t}) < t'$ for $f(\bar{t}) < \{t'\}$. We write $\text{SmtValid}(\Phi, p)$ for SMT validity of the implication $\Phi \Rightarrow p$.

Instantiation & Unfolding A term q is a (Ψ, Φ) -instance if there exists $f(\bar{t}) < \Phi$ such that:

- $\Psi(f) \equiv \lambda \bar{x}. \langle p_i \Rightarrow b_i \rangle$,
- $\text{SmtValid}(\Phi, p_i \ [\bar{t}/\bar{x}])$,
- $q \equiv (f(\bar{x}) = b_i) \ [\bar{t}/\bar{x}]$.

A set of terms Q is a (Ψ, Φ) -instance if every $q \in Q$ is an (Ψ, Φ) -instance. The *unfolding* of Ψ, Φ is the (finite) set of all (Ψ, Φ) -instances. Procedure $\text{Unfold}(\Psi, \Phi)$ shown in Figure 9 computes and returns the conjunction of Φ and the unfolding of Ψ, Φ . The following properties relate (Ψ, Φ) -instances to the semantics of λ^R and SMT validity. Let $R[e]$ denote the evaluation of e under the reflection environment R , i.e. $\emptyset[e] \doteq e$ and $(R, f : e_f)[e] \doteq R[\text{let rec } f = e_f \text{ in } e]$.

Unfold	: $(\Psi, \Phi) \rightarrow \Phi$
Unfold(Ψ, Φ)	= $\Phi \cup \bigcup_{f(\bar{t}) < \Phi} \text{Instantiate}(\Psi, \Phi, f, \bar{t})$
Instantiate(Ψ, Φ, f, \bar{t})	= $\{(\llbracket f(\bar{x}) \rrbracket = b_i) \llbracket \bar{t}/\bar{x} \rrbracket \mid (p_i \Rightarrow b_i) \in \bar{d}, \text{SmtValid}(\Phi, p_i \llbracket \bar{t}/\bar{x} \rrbracket)\}$
where	
$\lambda \bar{x}. \langle \bar{d} \rangle$	= $\Psi(f)$
PLE	: $(\Psi, \Phi, p) \rightarrow \text{Bool}$
PLE(Ψ, Φ, p)	= $\text{loop}(0, \Phi \cup \bigcup_{f(\bar{t}) < p} \text{Instantiate}(\Psi, \Phi, f, \bar{t}))$
where	
loop(i, Φ_i)	
SmtValid(Φ_i, p)	= <i>true</i>
$\Phi_{i+1} \subseteq \Phi_i$	= <i>false</i>
otherwise	= loop($i + 1, \Phi_{i+1}$)
where	
Φ_{i+1}	= $\Phi \cup \text{Unfold}(\Psi, \Phi_i)$

Fig. 9. Algorithm PLE: Proof by Logical Evaluation

LEMMA 6.1. For every $\Gamma \models R$, and $\theta \in \langle \Gamma \rangle$,

- **Sat-Inst** If $\llbracket e \rrbracket$ is a $(\llbracket R \rrbracket, \llbracket \Gamma \rrbracket)$ -instance, then $\theta \cdot R[e] \hookrightarrow^* \text{True}$.
- **SMT-Approx** If $\text{SmtValid}(\llbracket \Gamma \rrbracket, \llbracket e \rrbracket)$ then $\theta \cdot R[e] \hookrightarrow^* \text{True}$.
- **SMT-Inst** If q is a $(\llbracket R \rrbracket, \llbracket \Gamma \rrbracket)$ -instance and $\text{SmtValid}(\llbracket \Gamma \rrbracket \cup \{q\}, \llbracket e \rrbracket)$ then $\theta \cdot R[e] \hookrightarrow^* \text{True}$.

The Algorithm Figure 9 shows our proof search algorithm $\text{PLE}(\Psi, \Phi, p)$ which takes as input a set of *reflected definitions* Ψ , an *hypothesis* Φ , and a *goal* p . The PLE procedure recursively *unfolds* function application terms by invoking *Unfold* until either the goal can be proved using the unfolded instances (in which case the search returns *true*) or no new instances are generated by the unfolding (in which case the search returns *false*).

Soundness First, we prove the soundness of PLE.

THEOREM 6.2 (SOUNDNESS). If $\text{PLE}(\llbracket R \rrbracket, \llbracket \Gamma \rrbracket, \llbracket e \rrbracket)$ then $\forall \theta \in \langle \Gamma \rangle, \theta \cdot R[e] \hookrightarrow^* \text{True}$.

We prove Theorem 6.2 using the Lemma 6.1 that relates instantiation, SMT validity, and the exact semantics. Intuitively, PLE is sound as it reasons about a finite set of instances by *conservatively* treating all function applications as *uninterpreted* [Nelson 1980].

6.2 Completeness

Next, we show that our proof search is *complete* with respect to equational reasoning. We define a notion of equational proof $\Psi, \Phi \vdash t \twoheadrightarrow t'$ and prove that if there exists such a proof, then $\text{PLE}(\Psi, \Phi, t = t')$ is guaranteed to return *true*. To prove this theorem, we introduce the notion of *bounded unfolding* which corresponds to unfolding definitions n times. We will show that unfolding preserves congruences, and hence, that an equational proof exists iff the goal can be proved with *some* bounded unfolding. Thus, completeness follows by showing that the proof search procedure computes the limit (*i.e.* fixpoint) of the bounded unfolding. In § 6.3 we will show that the fixpoint is computable: there is an unfolding depth at which PLE reaches a fixpoint and hence terminates.

$$\begin{array}{c}
\frac{}{\Psi, \Phi \vdash t \rightarrow t} \text{EQ-REFL} \\
\frac{\Psi, \Phi \vdash t \rightarrow t'' \quad \Phi' = \text{Unfold}(\Psi, \Phi \cup \{v = t''\}) \quad \text{SmtValid}(\Phi', v = t')}{\Psi, \Phi \vdash t \rightarrow t'} \text{EQ-TRANS} \\
\frac{\Psi, \Phi \vdash t_1 \rightarrow t'_1 \quad \Psi, \Phi \vdash t_2 \rightarrow t'_2 \quad \text{SmtValid}(\Phi, t'_1 \bowtie t'_2)}{\Psi, \Phi \vdash t_1 \bowtie t_2} \text{EQ-PROOF}
\end{array}$$

Fig. 10. Equational Proofs: rules for equational reasoning

Bounded Unfolding For every Ψ, Φ and $0 \leq n$, the *bounded unfolding of depth n* is defined by:

$$\begin{aligned}
\text{Unfold}^*(\Psi, \Phi, 0) &\doteq \Phi \\
\text{Unfold}^*(\Psi, \Phi, n+1) &\doteq \Phi_n \cup \text{Unfold}(\Psi, \Phi_n) \quad \text{where } \Phi_n = \text{Unfold}^*(\Psi, \Phi, n)
\end{aligned}$$

That is, the unfolding at depth n essentially performs Unfold upto n times. The bounded-unfoldings yield a monotonically non-decreasing sequence of formulas such that if two consecutive bounded unfoldings coincide, then all subsequent unfoldings are the same.

LEMMA 6.3 (**MONOTONICITY**). $\forall 0 \leq n. \text{Unfold}^*(\Psi, \Phi, n) \subseteq \text{Unfold}^*(\Psi, \Phi, n+1)$.

LEMMA 6.4 (**FIXPOINT**). *Let $\Phi_i \doteq \text{Unfold}^*(\Psi, \Phi, i)$. If $\Phi_n = \Phi_{n+1}$ then $\forall n < m. \Phi_m = \Phi_n$.*

Uncovering Next we prove that every function application term that is *uncovered* by unfolding to depth n is congruent to a term in the n -depth unfolding.

LEMMA 6.5 (**UNCOVERING**). *Let $\Phi_n \equiv \text{Unfold}^*(\Psi, \Phi \cup \{v = t\}, n)$. If $\text{SmtValid}(\Phi_n, v = t')$ then for every $f(\bar{t}') < t'$ there exists $f(\bar{t}) < \Phi_n$ such that $\text{SmtValid}(\Phi_n, t_i = t'_i)$.*

We prove the above lemma by induction on n where the inductive step uses the following property of congruence closure, which itself is proved by induction on the structure of t' :

LEMMA 6.6 (**CONGRUENCE**). *If $\text{SmtValid}(\Phi \cup \{v = t\}, v = t')$ and $v \notin \Phi, t, t'$ then for every $f(\bar{t}') < t'$ there exists $f(\bar{t}) < \Phi, t$ such that $\text{SmtValid}(\Phi, t_i = t'_i)$.*

Unfolding Preserves Equational Links Next, we use the uncovering Lemma 6.5 and congruence to show that every *instantiation* that is valid after n steps is subsumed by the $n+1$ depth unfolding. That is, we show that every possible *link* in a possible equational chain can be proved equal to the source expression via bounded unfolding.

LEMMA 6.7 (**LINK**). *If $\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi \cup \{v = t\}, n), v = t')$ then $\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi \cup \{v = t\}, n+1), \text{Unfold}(\Psi, \Phi \cup \{v = t'\}))$.*

Equational Proof Figure 10 formalizes our rules for equational reasoning. Intuitively, there is an *equational proof* that $t_1 \bowtie t_2$ under Ψ, Φ written by the judgment $\Psi, \Phi \vdash t_1 \bowtie t_2$ if by some sequence of repeated function unfoldings, we can prove that t_1 and t_2 are respectively equal to t'_1 and t'_2 such that, $\text{SmtValid}(\Phi, t'_1 \bowtie t'_2)$ holds. Our notion of equational proofs adapts the idea of type level computation used in TT-based proof assistants to the setting of SMT-based reasoning, via the directional unfolding judgment $\Psi, \Phi \vdash t \rightarrow t'$. In the SMT-realm, the explicit notion of a normal or canonical form is converted to the implicit notion of the equivalence classes of the SMT solver's congruence closure procedure (post-unfolding).

Completeness of Bounded Unfolding Finally, we use the fact that unfolding preserves equational links to show that bounded unfolding is *complete* for equational proofs. That is, we prove by induction on the structure of the equational proof that whenever there is an *equational proof* of $t = t'$, there exists some bounded unfolding that suffices to prove the equality.

LEMMA 6.8. *If $\Psi, \Phi \vdash t \rightarrow t'$ then $\exists 0 \leq n. \text{SmtValid}(\text{Unfold}^*(\Psi, \Phi \cup \{v = t\}, n), v = t')$.*

PLE is a Fixpoint of Bounded Unfolding Next, we show that the proof search procedure PLE computes the least-fixpoint of the bounded unfolding and hence, returns *true* iff there exists *some* unfolding depth n at which the goal can be proved.

LEMMA 6.9 (FIXPOINT). *PLE($\Psi, \Phi, t = t'$) iff $\exists n. \text{SmtValid}(\text{Unfold}^*(\Psi, \Phi \cup \{v = t\}, n), v = t')$.*

The proof follows by observing that PLE($\Psi, \Phi, t = t'$) computes the *least-fixpoint* of the sequence $\Phi_i \doteq \text{Unfold}^*(\Psi, \Phi, i)$. Specifically, we can prove by induction on i that at each invocation of loop (i, Φ_i) in Figure 9, Φ_i is equal to $\text{Unfold}^*(\Psi, \Phi \cup \{v = t\}, i)$, which then yields the result.

Completeness of PLE By combining Lemma 6.9 and Lemma 6.7 we can show that PLE is complete, *i.e.* if there is an equational proof that $t \bowtie t'$ under Ψ, Φ , then PLE($\Psi, \Phi, t \bowtie t'$) returns *true*.

THEOREM 6.10 (COMPLETENESS). *If $\Psi, \Phi \vdash t \bowtie t'$ then PLE($\Psi, \Phi, t \bowtie t'$) = true.*

6.3 PLE Terminates

So far, we have shown that our proof search procedure PLE is both sound and complete. Both of these are easy to achieve simply by *enumerating* all possible instances and repeatedly querying the SMT solver. Such a monkeys-with-typewriters approach is rather impractical: it may never terminate. Fortunately, next, we show that in addition to being sound and complete with respect to equational proofs, if the hypotheses are transparent, then our proof search procedure always terminates. Next, we describe transparency and explain intuitively why PLE terminates. We then develop the formalism needed to prove the termination theorem 6.16.

Transparency An environment Γ is *inconsistent* if $\text{SmtValid}([\Gamma], \text{false})$. An environment Γ is *inhabited* if there exists some $\theta \in \langle \Gamma \rangle$. We say Γ is *transparent* if it is either inhabited or inconsistent. As an example of a *non-transparent* Φ_0 consider the predicate $\text{lenA } xs = 1 + \text{lenB } xs$, where lenA and lenB are both identical definitions of the list length function. Clearly there is no θ that causes the above predicate to evaluate to *true*. At the same time, the SMT solver cannot (using the decidable, quantifier-free theories) prove a contradiction as that requires induction over xs . Thus, non-transparent environments are somewhat pathological, and in practice, we only invoke PLE on transparent environments. Either the environment is inconsistent, *e.g.* when doing a proof-by-contradiction, or *e.g.* when doing a proof-by-case-analysis we can easily find suitable concrete values via random [Claessen and Hughes 2000] or SMT-guided generation [Seidel et al. 2015].

Challenge: Connect Concrete and Logical Semantics As suggested by its name, the PLE algorithm aims to lift the notion of evaluation or computations into the level of the refinement logic. Thus, to prove termination, we must connect the two different notions of evaluation, the *concrete* (operational) semantics and the *logical* semantics being used by PLE. This connection is trickier than appears at first glance. In the concrete realm totality ensures that every reflected function f will terminate when run on any *individual* value v . However, in the logical realm, we are working with *infinite* sets of values, compactly represented via logical constraints. In other words, the logical realm can be viewed (informally) as an *abstract interpretation*, of the concrete semantics. We must carefully argue that despite the *approximation* introduced by the logical abstraction, the abstract interpretation will also terminate.

Solution: Universal Abstract Interpretation We make this argument in three parts. First, we formalize how PLE performs computation at the logical level via *logical steps* and *logical traces*. We show (Lemma 6.13) that the logical steps form a so-called *universal* (or “*must*”) abstraction of the concrete semantics [Clarke et al. 1992; Cousot and Cousot 1977]. Second, we show that if PLE diverges, it is because it creates a strictly increasing infinite chain, $\text{Unfold}^*(\Psi, \Phi, 0) \subset \text{Unfold}^*(\Psi, \Phi, 1) \dots$ which corresponds to an *infinite logical trace*. Third, as the logical computation is universal abstraction we use inhabitation to connect the two realms, *i.e.* to show that an infinite logical trace corresponds to an infinite concrete trace. The impossibility of the latter must imply the impossibility of the former, *i.e.* PLE terminates. Next, we formalize the above to obtain Theorem 6.16.

Totality A function is *total* when its evaluation reduces to exactly one value. The totality of R can and is checked by refinement types (§ 4). Hence, for brevity, in the sequel we will *implicitly assume* that R is total under Γ .

Definition 6.11 (Total). Let $b \equiv \lambda \bar{x}. \langle \overline{\lfloor p \rfloor} \Rightarrow \overline{\lfloor e \rfloor} \rangle$. b is *total* under Γ and R if for all $\theta \in (\Gamma)$:

- (1) If $\theta \cdot R[p_i] \hookrightarrow^* \text{True}$ then $\exists v. \theta \cdot R[e_i] \hookrightarrow^* v$.
- (2) If $\theta \cdot R[p_i] \hookrightarrow^* \text{True}$ and $\theta \cdot \Psi[p_j] \hookrightarrow^* \text{True}$, then $i = j$.
- (3) There exists an i so that $\theta \cdot R[p_i] \hookrightarrow^* \text{True}$.

R is *total* under Γ if every $b \in \lfloor R \rfloor$ is total under Γ and R .

Subterm Evaluation As the reflected functions are total, the Church-Rosser theorem implies that evaluation order is not important. To prove termination, we require an evaluation strategy, *e.g.* CBV, in which if a reflected function’s guard is satisfied, then the evaluation of the corresponding function body requires evaluating *every subterm* inside the body. As $\text{DeIf}(\cdot)$ hoists *if*-expressions out of the body and into the top-level guards, the below fact follows from the properties of CBV:

LEMMA 6.12. Let $b \equiv \lambda \bar{x}. \langle \overline{\lfloor p \rfloor} \Rightarrow \overline{\lfloor e \rfloor} \rangle$, and $f \in R$. For every Γ, R , and $\theta \in (\Gamma)$, if $\theta \cdot R[p_i] \hookrightarrow^* \text{True}$ and $f(\overline{\lfloor e' \rfloor}) < \lfloor e_i \rfloor$ then $\theta \cdot R[e_i] \hookrightarrow^* C[f(\theta \cdot R[\overline{\lfloor e' \rfloor})]$.

Logical Step A pair $f(\bar{t}) \rightsquigarrow f'(\bar{t}')$ is a Ψ, Φ -*logical step* (abbrev. step) if

- $\Psi(f) \equiv \lambda \bar{x}. \langle \overline{\lfloor p \rfloor} \Rightarrow \overline{\lfloor b \rfloor} \rangle$,
- $\text{SmtValid}(\Phi \wedge Q, p_i)$ for some (Ψ, Φ) -instance Q ,
- $f'(\bar{t}') < b_i \lfloor \bar{t} / \bar{x} \rfloor$

Steps and Reductions Next, using Lemmas 6.12, 6.1, and the definition of logical steps, we show that every logical step corresponds to a *sequence* of steps in the concrete semantics:

LEMMA 6.13 (**STEP-REDUCTIONS**). If $f(\overline{\lfloor e \rfloor}) \rightsquigarrow f'(\overline{\lfloor e' \rfloor})$ is a *logical step* under $\lfloor R \rfloor, \lfloor \Gamma \rfloor$ and $\theta \in (\Gamma)$, then $f(\theta \cdot R[\overline{\lfloor e \rfloor}]) \hookrightarrow^* C[f(\theta \cdot R[\overline{\lfloor e' \rfloor})]$ for some context C .

Logical Trace A sequence $f_0(\bar{t}_0), f_1(\bar{t}_1), f_2(\bar{t}_2), \dots$ is a Ψ, Φ -*logical trace* (abbrev. trace) if $f_i(\bar{t}_i) \rightsquigarrow f_{i+1}(\bar{t}_{i+1})$ is a Ψ, Φ -step, for each i . Our termination proof hinges upon the following key result: inhabited environments only have *finite* logical traces. We prove this result by contradiction. Specifically, we show by Lemma 6.13 that an infinite $(\lfloor R \rfloor, \lfloor \Gamma \rfloor)$ -trace combined with fact that Γ is inhabited yields *at least one infinite concrete trace*, which contradicts totality. Hence, all the $(\lfloor R \rfloor, \lfloor \Gamma \rfloor)$ logical traces must be finite.

THEOREM 6.14 (**FINITE-TRACE**). If Γ is inhabited then every $(\lfloor R \rfloor, \lfloor \Gamma \rfloor)$ -trace is *finite*.

Ascending Chains and Traces If unfolding Ψ, Φ yields an infinite chain $\Phi_0 \subset \dots \subset \Phi_n \dots$, then Ψ, Φ has an infinite logical trace. We construct the trace by selecting, at level i , (*i.e.* in Φ_i), an application term $f_i(\bar{t}_i)$ that was created by unfolding an application term at level $i - 1$ (*i.e.* in Φ_{i-1}).

Benchmark	Common		Without PLE Search			With PLE Search		
	Impl (l)	Spec (l)	Proof (l)	Time (s)	SMT (q)	Proof (l)	Time (s)	SMT (q)
Arithmetic								
Fibonacci	7	10	38	2.74	129	16	1.92	79
Ackermann	20	73	196	5.40	566	119	13.80	846
Class Laws Fig 11								
Monoid	33	50	109	4.47	34	33	4.22	209
Functor	48	44	93	4.97	26	14	3.68	68
Applicative	62	110	241	12.00	69	74	10.00	1090
Monad	63	42	122	5.39	49	39	4.89	250
Higher-Order Properties								
Logical Properties	0	20	33	2.71	32	33	2.74	32
Fold Universal	10	44	43	2.17	24	14	1.46	48
Functional Correctness								
SAT-solver	92	34	0	50.00	50	0	50.00	50
Unification	51	60	85	4.77	195	21	5.64	422
Deterministic Parallelism								
Conc. Sets	597	329	339	40.10	339	229	40.70	861
n -body	163	251	101	7.41	61	21	6.27	61
Par. Reducers	30	212	124	6.63	52	25	5.56	52
Total	1176	1279	1524	148.76	1626	638	150.88	4068

Table 1. We report verification **Time** (in seconds, on a 2.3GHz Intel® Xeon® CPU E5-2699 v3 with 18 physical cores and 64GiB RAM.), the number of **SMT** queries and size of **Proofs** (in lines). The **Common** columns show sizes of common **Implementations** and **Specifications**. We separately consider proofs **Without** and **With PLE Search**.

LEMMA 6.15 (**ASCENDING CHAINS**). *Let $\Phi_i \doteq \text{Unfold}^*(\Psi, \Phi, i)$. If there exists an (infinite) ascending chain $\Phi_0 \subset \dots \subset \Phi_n \dots$ then there exists an (infinite) logical trace $f_0(t_0), \dots, f_n(t_n), \dots$*

Logical Evaluation Terminates Finally, we prove that the proof search procedure PLE terminates. If PLE loops forever, there must be an infinite strictly ascending chain of unfoldings Φ_i , and hence, by Lemma 6.15, an infinite logical trace, which, by Theorem 6.14, is impossible.

THEOREM 6.16 (**TERMINATION**). *If Γ is transparent then $\text{PLE}([R], [\Gamma], p)$ terminates.*

7 EVALUATION

We have implemented reflection and PLE in LIQUID HASKELL [Vazou et al. 2014a]. Table 1 summarizes our evaluation which aims to determine (1) the kinds of programs and properties that can be verified, (2) how PLE simplifies *writing* proofs, and (3) how PLE affects the verification time.

Benchmarks We summarize our benchmarks below, see Appendix (§ I, § J) for details.

- **Arithmetic** We proved arithmetic properties for the textbook Fibonacci function (c.f. § 2) and the 12 properties of the Ackermann function from [Tourlakis 2008].
- **Class Laws** We proved the monoid laws for the **Peano**, **Maybe** and **List** data types and the Functor, Applicative, and Monad laws, summarized in Figure 11, for the **Maybe**, **List** and **Identity** monads.
- **Higher Order Properties** We used natural deduction to prove textbook logical properties as in § 3. We combined natural deduction principles with PLE-search to prove universality of right-folds, as described in [Hutton 1999] and formalized in AGDA [Mu et al. 2009].

- **Functional Correctness** We proved correctness of a SAT solver and a unification algorithm as implemented in *Zombie* [Casinghino et al. 2014]. We proved that the SAT solver takes as input a formula f and either returns `Nothing` or an assignment that satisfies f , by reflecting the notion of satisfaction. Then, we proved that if the unification `unify s t` of two terms s and t returns a substitution su , then applying su to s and t yields identical terms. Note that, while the unification function can itself diverge, and hence, cannot be reflected, our method allows terminating and diverging functions to soundly coexist.
- **Deterministic Parallelism** Retrofitting verification onto an existing language with a mature parallel run-time allows us to create three deterministic parallelism libraries that, for the first time, verify implicit assumptions about associativity and ordering that are critical for determinism. First, we proved that the *ordering laws* hold for keys being inserted into `LVar`-style concurrent sets [Kuper et al. 2014]. Second, we used `monad-par` [Marlow et al. 2011] to implement an n -body simulation, whose correctness relied upon proving that a triple of `Real` (implementing) 3-d acceleration was a `Monoid`. Third, we built a `DPJ`-style [Bocchino et al. 2009] parallel-reducers library whose correctness relied upon verifying that the reduced arguments form a `CommutativeMonoid`, and which was the basis of a parallel array sum. Appendix J includes performance results.

Proof Effort We split the total lines of code of our benchmarks into three categories: **Spec** represents the refinement types that encode theorems, lemmas, and function *specifications*; **Impl** represents the rest of the Haskell code that defines executable functions; **Proofs** represent the sizes of the Haskell proof terms (*i.e.* functions returning `Prop`). Reflection and PLE are optionally enabled using pragmas; the latter is enabled either for a whole file/module, or per top-level function.

Results The main highlights of our evaluation are the following. (1) Reflection allows for the specification and verification of a wide variety of important properties of programs. (2) PLE drastically reduces the proof effort: by a factor of $2 - 5\times$ – shrinking the total lines of proof from 1524 to 638 – making it quite modest, about the size of the specifications of the theorems. Since PLE searches for equational properties, there are some proofs, that rarely occur in practice, that PLE cannot simplify, *e.g.* the logical properties from § 3. (3) PLE does not impose a performance penalty: even though proof search can make an order of magnitude many more SMT queries – increasing the total SMT queries from 1626 without PLE to 4068 with PLE – most of these queries are simple and it is typically *faster* to type-check the compact proofs enabled by PLE than it is to type-check the $2 - 5\times$ longer explicit proofs written by a human.

8 RELATED WORK

SMT-Based Verification SMT-solvers have been extensively used to automate program verification via Floyd-Hoare logics [Nelson 1980]. *LEON* introduces an SMT-based algorithm that is complete for catamorphisms (folds) over ADTs [Suter et al. 2010], and a semi-decision procedure that is guaranteed to find satisfying assignments (models) for queries over arbitrary recursive functions, if they exist [Suter et al. 2011]. Our work is inspired by *DAFNY*'s Verified Calculations [Leino and Polikarpova 2016] but differs in (1) our use of reflection instead of axiomatization, (2) our use of refinements to compose proofs, and (3) our use of PLE to automate reasoning about user-defined functions. *DAFNY* (and F^* [Swamy et al. 2016]) encode user-functions as axioms and use a fixed fuel to instantiate functions upto some fixed unfolding depth [Amin et al. 2014]. While the fuel-based approach is incomplete, even for equational or calculational reasoning, it may, although rare in practice, quickly time out after a fixed, small number of instantiations rather than perform an exhaustive proof search like PLE. Nevertheless, PLE demonstrates that it is possible to develop complete and practical algorithms for reasoning about user-defined functions.

Monoid (for Peano, Maybe, List)	Functor (for Maybe, List, Id)
Left Id. $\text{mempty } x \diamond \equiv x$	Id. $\text{fmap id } xs \equiv \text{id } xs$
Right Id. $x \diamond \text{mempty} \equiv x$	Distr. $\text{fmap } (g \circ h) \text{ } xs \equiv (\text{fmap } g \circ \text{fmap } h) \text{ } xs$
Assoc. $(x \diamond y) \diamond z \equiv x \diamond (y \diamond z)$	
Applicative (for Maybe, List, Id)	Monad (for Maybe, List, Id)
Id. $\text{pure id } \otimes v \equiv v$	Left Id. $\text{return } a \gg= f \equiv f a$
Comp. $\text{pure } (\circ) \otimes u \otimes v \otimes w \equiv u \otimes (v \otimes w)$	Right Id. $m \gg= \text{return} \equiv m$
Hom. $\text{pure } f \otimes \text{pure } x \equiv \text{pure } (f x)$	Assoc. $(m \gg= f) \gg= g \equiv m \gg= (\lambda x \rightarrow f x \gg= g)$
Inter. $u \otimes \text{pure } y \equiv \text{pure } (\$ y) \otimes u$	
Ord (for Int, Double, Either, (,))	Commutative Monoid (for Int, Double, (,))
Refl. $x \leq x$	Comm. $x \diamond y \equiv y \diamond x$
Antisym. $x \leq y \wedge y \leq x \implies x \equiv y$	(including Monoid laws)
Trans. $x \leq y \wedge y \leq z \implies x \leq z$	
Total. $x \leq y \vee y \leq x$	

Fig. 11. Summary of Verified Typeclass Laws

Proving Equational Properties Several authors have proposed tools for proving (equational) properties of (functional) programs. Systems of Sousa and Dillig [2016] and Asada et al. [2015] extend classical safety verification algorithms, respectively based on Floyd-Hoare logic and refinement types, to the setting of relational or k -safety properties that are assertions over k -traces of a program. Thus, these methods can automatically prove that certain functions are associative, commutative *etc.* but are restricted to first-order properties and are not programmer-extensible. Zeno [Sonnex et al. 2012] generates proofs by term rewriting and Halo [Vytiniotis et al. 2013] uses an axiomatic encoding to verify contracts. Both the above are automatic, but unpredictable and not programmer-extensible, hence, have been limited to far simpler properties than the ones checked here. HERMIT [Farmer et al. 2015] proves equalities by rewriting the GHC core language, guided by user specified scripts. Our proofs are Haskell programs, SMT solvers automate reasoning, and, importantly, we connect the validity of proofs with the semantics of the programs.

Dependent Types in Programming Integration of dependent types into Haskell has been a long standing goal [Eisenberg and Stolarek 2014] that dates back to Cayenne [Augustsson 1998], a Haskell-like, fully dependent type language with undecidable type checking. Our approach differs significantly in that reflection and PLE use SMT-solvers to drastically simplify proofs over decidable theories. Zombie [Sjöberg and Weirich 2015] investigates the design of a dependently typed language where SMT-style congruence closure is used to reason about the equality of terms. However, Zombie explicitly eschews type-level computation as the authors write “equalities that follow from β -reduction” are “incompatible with congruence closure”. Due to this incompleteness, the programmer must use explicit `join` terms to indicate where normalization should be triggered, even so, equality checking is based on fuel, hence, is incomplete.

Proof Assistants Reflection shows how to retrofit deep specification and verification in the style of AGDA [Norell 2007], COQ [Bertot and Castéran 2004] and ISABELLE [Nipkow et al. 2002] into existing languages via refinement typing and PLE shows how type-level computation can be made compatible with SMT solvers’ native theory reasoning yielding a powerful new way to automate proofs (§ 2.5). Mature proof assistants like AGDA, COQ, and ISABELLE have two clear advantages over our approach: they emit certificates, so they rely on a small trusted computing base, and they have decades-worth of tactics, libraries and proof scripts that enable large scale

proof engineering. Some tactics even enable embedding of SMT-based proof search heuristics, e.g. SLEDGEHAMMER [Blanchette et al. 2011], that is widely used in ISABELLE. However, this search does not have the completeness guarantees of PLE. The issue of extracting checkable certificates from SMT solvers is well understood [Chen et al. 2010; Necula 1997] and easy to extend to our setting. However, the question of extending SMT-based verifiers with tactics and scriptable proof search, and more generally, incorporating *interactivity* in the style of proof-assistants, perhaps enhanced by proof-completion hints remains an interesting direction for future work.

9 CONCLUSIONS AND FUTURE WORK

Thus, our results identify a new design for deductive verifiers wherein: (1) via Refinement Reflection, we can encode natural deduction proofs as SMT-checkable refinement typed programs; (2) via Proof by Logical Evaluation we can combine the complementary strengths of SMT- (*i.e.*, decision procedures) and TT- based approaches (*i.e.*, type-level computation) to obtain completeness guarantees when verifying properties of user-defined functions. However, the increased automation of SMT and proof-search can sometimes make it harder for a user to debug *failed* proofs. In future work, it would be interesting to investigate how to add interactivity to SMT based verifiers, in the form of tactics and scripts or algorithms for synthesizing proof hints, and by designing new ways to explain and fix refinement type errors.

REFERENCES

- N. Amin, K. R. M. L., and T. Rompf. 2014. Computing with an SMT Solver. In *TAP*.
- A. Appel. 2016. *Verified Functional Algorithms*. <https://www.cs.princeton.edu/~appel/vfa/Perm.html>.
- K. Asada, R. Sato, and N. Kobayashi. 2015. Verifying Relational Properties of Functional Programs by First-Order Refinement. In *PEPM*.
- L. Augustsson. 1998. Cayenne - a Language with Dependent Types.. In *ICFP*.
- C. Barrett, A. Stump, and C. Tinelli. 2010. The SMT-LIB Standard: Version 2.0.
- J. Bengtson, K. Bhargavan, C. Fournet, A.D. Gordon, and S. Maffei. 2008. Refinement Types for Secure Implementations. In *CSF*.
- Y. Bertot and P. Castéran. 2004. *Coq'Art: The Calculus of Inductive Constructions*. Springer Verlag.
- R. S. Bird. 1989. Algebraic Identities for Program Calculation. In *The Computer Journal*.
- J. C. Blanchette, S. Böhme, and L. C. Paulson. 2011. Extending Sledgehammer with SMT Solvers. In *CADE*.
- Jr. Bocchino, L. Robert, V. S. Adve, S. V. Adve, and M. Snir. 2009. Parallel Programming Must Be Deterministic by Default. In *HotPar*.
- C. Casinghino, V. Sjöberg, and S. Weirich. 2014. Combining proofs and programs in a dependently typed language. In *POPL*.
- J. Chen, R. Chugh, and N. Swamy. 2010. Type-preserving Compilation of End-to-end Verification of Security Enforcement. In *PLDI*.
- K. Claessen and J. Hughes. 2000. QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs. In *ICFP*.
- E. M. Clarke, O. Grumberg, and D.E. Long. 1992. Model checking and abstraction. In *POPL*.
- S. Collange, D. Defour, S. Graillat, and R. Iakymchuk. 2014. Full-Speed Deterministic Bit-Accurate Parallel Floating-Point Summation on Multi- and Many-Core Architectures. (2014). <https://hal.archives-ouvertes.fr/hal-00949355> working paper or preprint.
- J. Condit, M. Harren, Z. R. Anderson, D. Gay, and G. C. Necula. 2007. Dependent Types for Low-Level Programming. In *ESOP*.
- R. L. Constable and S. F. Smith. 1987. Partial Objects In Constructive Type Theory. In *LICS*.
- P. Cousot and R. Cousot. 1977. Abstract Interpretation: A Unified Lattice Model for the Static Analysis of Programs. In *POPL*.
- E. W. Dijkstra. 1975. Guarded Commands, Nondeterminacy and Formal Derivation of Programs. In *Communications of the ACM*.
- E. W. Dijkstra. 1976. *A Discipline of Programming*. Prentice-Hall.
- R. A. Eisenberg and J. Stolarek. 2014. Promoting functions to type families in Haskell. In *Haskell*.
- A. Farmer, N. Sulthorpe, and A. Gill. 2015. Reasoning with the HERMIT: Tool Support for Equational Reasoning on GHC Core Programs. In *Haskell*.
- G. Gentzen. 1935. Investigations into Logical Deduction. In *American Philosophical Quarterly*.

- W. A. Howard. 1980. The formulae-as-types notion of construction. In *Essays on Combinatory Logic, Lambda Calculus and Formalism*.
- G. Hutton. 1999. A tutorial on the universality and expressiveness of fold. *J. Functional Programming* (1999).
- A. M. Kent, D. Kempe, and S. Tobin-Hochstadt. 2016. Occurrence typing modulo theories. In *PLDI*.
- K. W. Knowles and C. Flanagan. 2010. Hybrid type checking. In *TOPLAS*.
- L. Kuper, A. Turon, N. R. Krishnaswami, and R. R. Newton. 2014. Freeze after writing: quasi-deterministic parallel programming with LVars. In *POPL*.
- K. R. M. Leino. 2010. Dafny: An Automatic Program Verifier for Functional Correctness. In *LPAR*.
- K. R. M. Leino and C. Pit-Claudel. 2016. Trigger selection strategies to stabilize program verifiers. In *CAV*.
- K. R. M. Leino and N. Polikarpova. 2016. Verified Calculations. In *VSTTE*.
- R. Leino. 2016. Dafny. (2016). <https://github.com/Microsoft/dafny/blob/master/Test/dafny0/Fuel.dfy>.
- S. Marlow, R. Newton, and S. Peyton-Jones. 2011. A Monad for Deterministic Parallelism. In *Haskell*.
- S. C. Mu, H. S. Ko, and P. Jansson. 2009. Algebra of Programming in Agda: Dependent Types for Relational Program Derivation. In *J. Funct. Program.*
- G. C. Necula. 1997. Proof carrying code. In *POPL*.
- C. G. Nelson. 1980. *Techniques for Program Verification*. Ph.D. Dissertation. Stanford University.
- T. Nipkow, L.C. Paulson, and M. Wenzel. 2002. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*.
- U. Norell. 2007. *Towards a practical programming language based on dependent type theory*. Ph.D. Dissertation. Chalmers.
- X. Ou, G. Tan, Y. Mandelbaum, and D. Walker. 2004. Dynamic Typing with Dependent Types. In *IFIP TCS*.
- J. C. Reynolds. 1972. Definitional interpreters for higher-order programming languages. In *ACM National Conference*.
- S. R. Della Rocca and L. Paolini. 2004. *The Parametric Lambda Calculus, A Metamodel for Computation*.
- P. Rondon, M. Kawaguchi, and R. Jhala. 2008. Liquid Types. In *PLDI*.
- P. Rondon, M. Kawaguchi, and R. Jhala. 2010. Low-Level Liquid Types. In *POPL*.
- J. Rushby, S. Owre, and N. Shankar. 1998. Subtypes for Specifications: Predicate Subtyping in PVS. *TSE*.
- E. L. Seidel, N. Vazou, and R. Jhala. 2015. Type Targeted Testing. In *ESOP*.
- V. Sjöberg and S. Weirich. 2015. Programming Up to Congruence. In *POPL*.
- W. Sonnex, S. Drossopoulou, and S. Eisenbach. 2012. Zeno: An Automated Prover for Properties of Recursive Data Structures. In *TACAS*.
- M. Sousa and I. Dillig. 2016. Cartesian hoare logic for verifying k-safety properties. In *PLDI*.
- Martin Sulzmann, Gregory J. Duck, Simon P. Jones, and Peter J. Stuckey. 2006. Understanding Functional Dependencies via Constraint Handling Rules. *Journal of Functional Programming* (2006).
- P. Suter, M. Dotta, and V. Kuncak. 2010. Decision Procedures for Algebraic Data Types with Abstractions. In *POPL*.
- P. Suter, A. Sinan Köksal, and V. Kuncak. 2011. Satisfiability Modulo Recursive Programs. In *SAS*.
- N. Swamy, C. Hrițcu, C. Keller, A. Rastogi, A. Delignat-Lavaud, S. Forest, K. Bhargavan, C. Fournet, P. Y. Strub, M. Kohlweiss, J. K. Zinzindohoue, and S. Zanella-Béguélin. 2016. Dependent Types and Multi-Monadic Effects in F*. In *POPL*.
- G. Tourlakis. 2008. Ackermann's Function. (2008). <http://www.cs.yorku.ca/~gt/papers/Ackermann-function.pdf>.
- N. Vazou, P. Rondon, and R. Jhala. 2013. Abstract Refinement Types. In *ESOP*.
- N. Vazou, E. L. Seidel, R. Jhala, D. Vytiniotis, and S. L. Peyton-Jones. 2014a. Refinement Types for Haskell. In *ICFP*.
- N. Vazou, E. L. Seidel, R. Jhala, D. Vytiniotis, and S. L. Peyton-Jones. 2014b. Technical Report: Refinement Types for Haskell. (2014). <http://goto.ucsd.edu/~nvazou/icfp14/haskell-refinements-techrep.pdf>.
- N. Vazou, A. Tondwalkar, V. Choudhury, R. G. Scott, R. R. Newton, P. Wadler, and R. Jhala. 2017. Extended Version: Refinement Reflection: Complete Verification with SMT. (2017). <https://nikivazou.github.io/static/pop18/extended-refinement-reflection.pdf>
- P. Vekris, B. Cosman, and R. Jhala. 2016. Refinement types for TypeScript. In *PLDI*.
- D. Vytiniotis, S. L. Peyton-Jones, K. Claessen, and D. Rosén. 2013. HALO: Haskell to Logic through Denotational Semantics. In *POPL*.
- P. Wadler. 1987. A Critique of Abelson and Sussman or Why Calculating is Better Than Scheming. In *SIGPLAN Not.*
- P. Wadler. 2015. Propositions As Types. In *Communications of the ACM*.
- H. Xi and F. Pfenning. 1998. Eliminating Array Bound Checking Through Dependent Types. In *PLDI*.

A PROOF OF MAPFUSION WITHOUT PLE

```

map_fusion f g []
= (map f . map g) []
=. map f (map g [])
=. map f []
=. map (f . g) []
** QED

map_fusion f g (C x xs)
= map (f . g) (x : xs)
=. ((f . g) x) : (map (f . g) xs)
=. ((f . g) x) : (((map f) . (map g)) xs)
  ? map_fusion f g xs
=. ((f . g) x) : (map f (map g xs))
=. (f (g x)) : (map f (map g xs))
=. map f ((g x) : (map g xs))
=. map f ((g x) : (map g xs))
=. map f (map g (x : xs))
=. map f ((map g) (x : xs))
=. ((map f) . (map g)) (x : xs)
** QED

```

B PROOFS FOR PLE

Proofs for § 6

PROOF. (Of lemma 6.5) We prove the result by induction on n .

Case $n = 0$: Immediate as $t \equiv t'$.

Case $n = k + 1$: Consider any t' such that $\text{SmtValid}(\Phi_{k+1}, v = t')$. By definition $\Phi_{k+1} = \text{Unfold}(\Psi, \Phi_k)$, hence $\text{SmtValid}(\text{Unfold}(\Psi, \Phi_k), v = t, v = t')$. Consider any $f(\bar{t}') < t'$; Lemma 6.6 completes the proof. \square

PROOF. (Of lemma 6.6) Proof by induction on the structure of t' : Case $t' \equiv x$: There are no subterms, hence immediate. Case $t' \equiv c$: There are no subterms, hence immediate. Case $t' \equiv f(\bar{t}')$: Consider the *last link* in Φ connecting the equivalence class of v (and t) to t' . Suppose the last link is a *congruence link* of the form $t = t'$ where $t \equiv f(\bar{t})$ and $\text{SmtValid}(\Phi, t = t')$. Then $f(\bar{t}) < \Phi, t$ and we are done. Suppose instead, the last link is an *equality link* in Φ of the form $z = f(\bar{t}')$. In this case, $f(\bar{t}') < \Phi$ and again, we are done. \square

PROOF. (Of lemma 6.7)

Let $G_k \doteq \text{Unfold}^*(\Psi, \Phi, k)$. Let us assume that

$$\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi, v = t, n), v = t') \quad (3)$$

Consider any instance

$$q \equiv f(\bar{t}') = b_i \left[\frac{\bar{t}'}{\bar{x}} \right] \text{ in } \text{Unfold}(\Psi, \Phi \wedge v = t') \quad (4)$$

By the definition of Unfold , we have

$$f(\bar{t}') < \Phi \wedge v = t' \text{ such that } \text{SmtValid}(\Phi, p_i \left[\frac{\bar{t}'}{\bar{x}} \right]) \quad (5)$$

By (3) and Lemma 6.5 there exists $f(\bar{t}) < \Phi_n$ such that

$$\text{SmtValid}(\Phi_n, t = t')$$

As $\Phi \subseteq \Phi_n$ and (5), by congruence

$$\text{SmtValid}(\Phi_n, p_i \left[\frac{\bar{t}}{\bar{x}} \right])$$

Hence, the instance

$$f(\bar{t}) = b_i \left[\frac{\bar{t}}{\bar{x}} \right] \text{ is in } \Phi_{n+1}$$

That is

$$\text{SmtValid}(\Phi_{n+1}, t = t' \wedge f(\bar{t}) = b_i [t/x])$$

And so by congruence closure

$$\text{SmtValid}(\Phi_{n+1}, q)$$

As the above holds for every instance, we have

$$\text{SmtValid}(\Phi_{n+1}, \text{Unfold}(\Psi, \Phi \wedge v = t'))$$

□

PROOF. (Of lemma 6.8) The proof follows by induction on the structure of $\Psi, \Phi \vdash t \rightarrow t'$.

Base Case EQ-REFL: Follows immediately as $t \equiv t'$.

Inductive Case EQ-TRANS

In this case, there exists t'' such that

$$\Psi, \Phi \vdash t \rightarrow t'' \tag{6}$$

$$\text{SmtValid}(\text{Unfold}(\Psi, \Phi \wedge v = t''), v = t') \tag{7}$$

By the induction hypothesis (6) implies there exists $0 \leq n$ such that

$$\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi \wedge v = t, n), v = t'')$$

By Lemma 6.7 we have

$$\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi, v = t, n + 1), \text{Unfold}(\Psi, \Phi \wedge v = t''))$$

Thus, by (7) and modus ponens we get

$$\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi, v = t, n + 1), v = t')$$

□

PROOF. (Of Lemma 6.9) Let $\Phi' = \Phi \wedge v = t$.

Case \Rightarrow : Assume that $\text{PLE}(\Psi, \Phi, t = t')$. That is, at some iteration i we have $\text{SmtValid}(\Phi_i, v = t')$, i.e. by (6.2) we have $\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi', i), v = t')$.

Case \Leftarrow : Pick the smallest n such that $\text{SmtValid}(\text{Unfold}^*(\Psi, \Phi', n), v = t')$. Using Lemmas 6.3 and 6.4 we can then show that for all $0 \leq k < n$, we have

$$\text{Unfold}^*(\Psi, \Phi', k) \subset \text{Unfold}^*(\Psi, \Phi', k + 1)$$

and

$$\text{Unfold}^*(\Psi, \Phi', k) \not\vdash v = t'$$

Hence, after n iterations of the recursive loop, $\text{PLE}(\Psi, \Phi, t = t')$, returns *true*. □

Steps and Values Next, we show that if $f(\bar{y}) \rightsquigarrow t'$ is a logical step under an Γ that is inhabited by θ then $f(\bar{y})$ reduces to a value under θ . The proof follows by observing that if Γ is inhabited by θ , and a particular step is possible, then the guard corresponding to that step must also be true under θ and hence, by totality, the function must reduce to a value under the given store.

LEMMA B.1 (**STEP-VALUE**). *If $\theta \in \langle \Gamma \rangle$ and $f(\bar{y}) \rightsquigarrow t'$ is a $[R], [\Gamma]$ step then $R[\theta \cdot f(\bar{y})] \hookrightarrow^* v$.*

PROOF. (Of Lemma B.1)

$$\text{Assume that } \theta \in \langle \Gamma \rangle \quad (8)$$

$$\text{Let } \theta^* \doteq \theta[\bar{x} \mapsto \theta \cdot \bar{y}] \quad (9)$$

$$\Psi(f) \doteq \lambda \bar{x}. \langle p \Rightarrow b \rangle \quad (10)$$

As $f(\bar{y}) \rightsquigarrow t'$ is a $[R]\Gamma$ step, for some i , $[R]$ -instance Q we have

$$\text{SmtValid}([\Gamma] \wedge Q, p_i [\bar{y}/\bar{x}])$$

$$\text{Hence, by (8) and Lemma 6.1 } \theta \cdot R[p_i [\bar{y}/\bar{x}]] \hookrightarrow^* \text{True} \quad (11)$$

$$\text{As } \theta^* \cdot p_i \equiv \theta \cdot p_i [\bar{y}/\bar{x}] \quad (12)$$

$$\text{The fact (11) yields } \theta^* \cdot R[p_i] \hookrightarrow^* \text{True}$$

$$\text{By the Totality Assumption 6.11 } R[\theta^* \cdot f(\bar{x})] \hookrightarrow^* v$$

$$\text{That is } R[\theta \cdot f(\bar{y})] \hookrightarrow^* v \quad (13)$$

$$(14)$$

□

Divergence A closed term t *diverges under R* if there is no v such that $R[t] \hookrightarrow^* v$.

LEMMA B.2 (**DIVERGENCE**). *If $\forall 0 \leq i$ we have $R[t_i] \hookrightarrow^* C[t_{i+1}]$ then t_0 diverges under Ψ .*

PROOF. (Of Theorem 6.14)

$$\text{Assume that } \theta \in \langle \Gamma \rangle \quad (15)$$

$$\text{and assume an infinite } [R], [\Gamma] \text{ trace: } f_0(\bar{t}_0), f_1(\bar{t}_1), \dots \quad (16)$$

$$\text{Where additionally } \bar{t}_0 \equiv \bar{x}_0 \quad (17)$$

$$\text{Define } t_i^* \equiv \theta \cdot f_i(\bar{t}_i) \quad (18)$$

$$\text{By Lemma 6.13, for every } i \in \mathbb{N} \quad R[t_i^*] \hookrightarrow^* C_i[t_{i+1}^*]$$

$$\text{Hence, by Lemma B.2 } t_0^* \text{ diverges under } \Psi$$

$$\text{i.e., by (17, 18) } \theta \cdot f_0(\bar{x}_0) \text{ diverges under } \Psi \quad (19)$$

$$\text{But by (15) and Lemma B.1 } R[\theta \cdot f_0(\bar{x}_0)] \hookrightarrow^* v \text{ contradicting (19)}$$

Hence, assumption (16) cannot hold, i.e. all the Ψ, Φ symbolic traces must be finite. □

PROOF. (Of Theorem 6.16) As Φ is transparent, there are two cases.

Case: Γ is inconsistent.

$$\text{By definition of inconsistency } \text{SmtValid}([\Gamma], \text{false})$$

$$\text{Hence } \text{SmtValid}([\Gamma], p)$$

$$\text{That is } \text{PLE}([R], [\Gamma], p) \text{ terminates immediately.}$$

Case: Γ is inhabited.

$$\text{That is, exists } \theta \text{ s.t. } \theta \in \langle \Gamma \rangle \quad (20)$$

Suppose that $\text{PLE}(\llbracket R \rrbracket, \llbracket \Gamma \rrbracket, p)$ does not terminate.

$$\text{That is, there is an infinitely increasing chain: } \Phi_0 \subset \dots \subset \Phi_n \dots \quad (21)$$

By Lemma 6.15 $\llbracket R \rrbracket, \llbracket \Gamma \rrbracket$ has an infinite trace

Which, by (20) contradicts Theorem 6.14. Thus, (21) is impossible, *i.e.* $\text{PLE}(\llbracket R \rrbracket, \llbracket \Gamma \rrbracket, p)$ terminates. \square

C PROOF OF SECTION: EMBEDDING NATURAL DEDUCTION WITH REFINEMENT TYPES

LEMMA C.1 (VALIDITY). *If there exists $e \in \langle \phi \rangle$ then ϕ is valid.*

PROOF. We prove the lemma by case analysis in the shape of ϕ .

- $\phi \equiv \{p\}$. Since the set $\langle \{p\} \rangle = \{e \mid p \hookrightarrow^* \text{True}\}$ is not empty, then $p \hookrightarrow^* \text{True}$.
- $\phi \equiv \phi_1 \rightarrow \phi_2$. By assumption, there exists an expressions f so that $\forall e_x \in \langle \phi_1 \rangle, f e_x \in \langle \phi_2 \rangle$. So, if there exists an expression $e_1 \in \langle \phi_1 \rangle$ that makes ϕ_1 valid then $f e_1$ makes ϕ_2 valid.
- $\phi \equiv \phi \rightarrow \text{False}$. By assumption, there exists an expressions f so that $\forall e_x \in \langle \phi \rangle, f e_x \in \langle \{\text{False}\} \rangle$. So, if there exists an expression $e_1 \in \langle \phi \rangle$ that makes ϕ valid then $f e_1$ makes $\{\text{False}\}$ valid, which is impossible, thus ϕ cannot be valid.
- $\phi \equiv (\phi_1, \phi_2)$. If there exists a total expression $e \in \langle \phi \rangle$ then e evaluates to (e_1, e_2) .
- $\phi \equiv \text{Either } \phi_1 \phi_2$. If there exists a total expression $e \in \langle \phi \rangle$ then e evaluates to either Left e' or Right e' .
- $\phi \equiv x : \tau \rightarrow \phi$. By assumption, there exists an expressions f so that $\forall e_x \in \langle \tau \rangle, f e_x \in \langle \phi[x/e_x] \rangle$. So, if there exists an expression $e_1 \in \langle \tau \rangle$ then $f e_1$ makes $\phi[x/e_1]$ valid.
- $\phi \equiv (x :: \tau, \phi)$. By assumption, there exists an expressions p that evaluates to a pair (e_x, e_y) so that $e_x \in \langle \tau \rangle$ and $e_y \in \langle \phi[x/e_x] \rangle$.

\square

THEOREM C.2 (VALIDITY). *If $\emptyset; \emptyset \vdash e : \phi$ then ϕ is valid.*

PROOF. By direct implication of Lemma C.1 and soundness of λ^R (Theorem 4.1). \square

D REFINEMENT REFLECTION: λ^R : EXTENDED VERSION WITH PROOFS

Our first step towards formalizing refinement reflection is a core calculus λ^R with an *undecidable* type system based on denotational semantics. We show how the soundness of the type system allows us to *prove theorems* using λ^R .

D.1 Syntax

Figure 12 summarizes the syntax of λ^R , which is essentially the calculus λ^U [Vazou et al. 2014a] with explicit recursion and a special `reflect` binding form to denote terms that are reflected into the refinement logic. In λ^R refinements r are arbitrary expressions e (hence $r ::= e$ in Figure 12). This choice allows us to prove preservation and progress, but renders typechecking undecidable. In § F we will see how to recover decidability by soundly approximating refinements.

The syntactic elements of λ^R are layered into primitive constants, values, expressions, binders and programs.

Operators	$\odot ::= = <$
Constants	$c ::= \wedge ! \odot +, -, \dots$ $ \text{True} \text{False} 0, 1, -1, \dots$
Values	$w ::= c \lambda x. e D \bar{w}$
Expressions	$e ::= w x e e$ $ \text{case } x = e \text{ of } \{D \bar{x} \rightarrow e\}$
Binders	$b ::= e \text{let rec } x : \tau = b \text{ in } b$
Program	$p ::= b \text{reflect } x : \tau = e \text{ in } p$
Basic Types	$B ::= \text{Int} \text{Bool} T$
Refined Types	$\tau ::= \{v : B \{\} \} x : \tau \rightarrow \tau$

Fig. 12. Syntax of λ^R **Contexts**

C

$C ::=$	\bullet
	$ C e c C D \bar{e} C \bar{e}$
	$ \text{case } y = C \text{ of } \{D_i \bar{x}_i \rightarrow e_i\}$

Reductions $p \hookrightarrow p'$

$C[p]$	$\hookrightarrow C[p'], \text{ if } p \hookrightarrow p'$
$c v$	$\hookrightarrow \delta(c, v)$
$(\lambda x. e) e'$	$\hookrightarrow e[e'/x]$
$\text{case } y = D_j \bar{e} \text{ of } \{D_i \bar{x}_i \rightarrow e_i\}$	$\hookrightarrow e_j [D_j \bar{e}/y] [\bar{e}/\bar{x}_i]$
$\text{reflect } x : \tau = e \text{ in } p$	$\hookrightarrow p[\text{fix } (\lambda x. e)/x]$
$\text{let rec } x : \tau = b_x \text{ in } b$	$\hookrightarrow b[\text{fix } (\lambda x. b_x)/x]$
$\text{fix } p$	$\hookrightarrow p(\text{fix } p)$

Fig. 13. Operational Semantics of λ^R

Constants The primitive constants of λ^R include all the primitive logical operators \oplus , here, the set $\{=, <\}$. Moreover, they include the primitive booleans True, False, integers $-1, 0, 1$, etc., and logical operators $\wedge, \vee, !$, etc..

Data Constructors We encode data constructors as special constants. For example the data type $[\text{Int}]$, which represents finite lists of integers, has two data constructors: $[\]$ (“nil”) and $:$ (“cons”).

Values & Expressions The values of λ^R include constants, λ -abstractions $\lambda x. e$, and fully applied data constructors D that wrap values. The expressions of λ^R include values and variables x , applications $e e$, and case expressions.

Binders & Programs A *binder* b is a series of possibly recursive let definitions, followed by an expression. A *program* p is a series of reflect definitions, each of which names a function that can be reflected into the refinement logic, followed by a binder. The stratification of programs via binders is required so that arbitrary recursive definitions are allowed but cannot be inserted into the logic via refinements or reflection. (We *can* allow non-recursive let binders in e , but omit them for simplicity.)

D.2 Operational Semantics

Figure 12 summarizes the small step contextual β -reduction semantics for λ^R . We write $e \hookrightarrow^j e'$ if there exist e_1, \dots, e_j such that e is e_1 , e' is e_j and $\forall i, j, 1 \leq i < j$, we have $e_i \hookrightarrow e_{i+1}$. We write $e \hookrightarrow^* e'$ if there exists some finite j such that $e \hookrightarrow^j e'$. We define \approx_β to be the reflexive, symmetric, transitive closure of \hookrightarrow .

Constants Application of a constant requires the argument be reduced to a value; in a single step the expression is reduced to the output of the primitive constant operation. For example, consider $=$, the primitive equality operator on integers. We have $\delta(=, n) \doteq =_n$ where $\delta(=, m)$ equals True iff m is the same as n . We assume that the equality operator is defined for all values, and, for functions, is defined as extensional equality. That is, for all f and f' we have $(f = f') \hookrightarrow \text{True}$ iff $\forall v. f v \approx_\beta f' v$. We assume source terms only contain implementable equalities over non-function types; the above only appears in refinements and allows us to state and prove facts about extensional equality § H.2.

D.3 Types

λ^R types include basic types, which are refined with predicates, and dependent function types. Basic types B comprise integers, booleans, and a family of data-types T (representing lists, trees etc..). For example the data type $[Int]$ represents lists of integers. We refine basic types with predicates (boolean valued expressions e) to obtain basic refinement types $\{v : B \mid e\}$. Finally, we have dependent function types $x : \tau_x \rightarrow \tau$ where the input x has the type τ_x and the output τ may refer to the input binder x . We write B to abbreviate $\{v : B \mid \text{True}\}$, and $\tau_x \rightarrow \tau$ to abbreviate $x : \tau_x \rightarrow \tau$ if x does not appear in τ . We use r to refer to refinements.

Denotations Each type τ denotes a set of expressions $\llbracket \tau \rrbracket$, that are defined via the dynamic semantics [Knowles and Flanagan 2010]. Let $\text{shape}(\tau)$ be the type we get if we erase all refinements from τ and $e : \text{shape}(\tau)$ be the standard typing relation for the typed lambda calculus. Then, we define the denotation of types as:

$$\begin{aligned} \llbracket \{x : B \mid r\} \rrbracket &\doteq \{e \mid e : B, \text{ if } e \hookrightarrow^* w \text{ then } r[x/w] \hookrightarrow^* \text{True}\} \\ \llbracket x : \tau_x \rightarrow \tau \rrbracket &\doteq \{e \mid e : \text{shape}(\tau_x \rightarrow \tau), \forall e_x \in \llbracket \tau_x \rrbracket. e e_x \in \llbracket \tau[x/e_x] \rrbracket\} \end{aligned}$$

Constants For each constant c we define its type $\text{prim}(c)$ such that $c \in \llbracket \text{prim}(c) \rrbracket$. For example,

$$\begin{aligned} \text{prim}(3) &\doteq \{v : \text{Int} \mid v = 3\} \\ \text{prim}(+) &\doteq x : \text{Int} \rightarrow y : \text{Int} \rightarrow \{v : \text{Int} \mid v = x + y\} \\ \text{prim}(\leq) &\doteq x : \text{Int} \rightarrow y : \text{Int} \rightarrow \{v : \text{Bool} \mid v \Leftrightarrow x \leq y\} \end{aligned}$$

So, by definition we get the constant typing lemma

LEMMA D.1. [Constant Typing] Every constant $c \in \llbracket \text{prim}(c) \rrbracket$.

Thus, if $\text{prim}(c) \doteq x : \tau_x \rightarrow \tau$, then for every value $w \in \llbracket \tau_x \rrbracket$, we require $\delta(c, w) \in \llbracket \tau[x/w] \rrbracket$.

D.4 Refinement Reflection

The simple, but key idea in our work is to *strengthen* the output type of functions with a refinement that *reflects* the definition of the function in the logic. We do this by treating each reflect-binder: reflect $f : \tau = e$ in p as a let rec-binder: let rec $f : \text{Reflect}(\tau, e) = e$ in p during type checking (rule T-REFL in Figure 7).

Reflection We write $\text{Reflect}(\tau, e)$ for the *reflection* of term e into the type τ , defined by strengthening τ as:

$$\begin{aligned} \text{Reflect}(\{v : B \mid r\}, e) &\doteq \{v : B \mid r \wedge v = e\} \\ \text{Reflect}(x : \tau_x \rightarrow \tau, \lambda y. e) &\doteq x : \tau_x \rightarrow \text{Reflect}(\tau, e[y/x]) \end{aligned}$$

As an example, recall from § 2 that the **reflect** fib strengthens the type of fib with the reflected refinement fibP.

Consequences for Verification Reflection has two consequences for verification. First, the reflected refinement is *not trusted*; it is itself verified (as a valid output type) during type checking. Second, instead of being tethered to quantifier instantiation heuristics or having to program “triggers” as in Dafny [Leino 2010] or F* [Swamy et al. 2016] the programmer can predictably “unfold” the definition of the function during a proof simply by “calling” the function, which we have found to be a very natural way of structuring proofs § 7.

D.5 Refining & Reflecting Data Constructors with Measures

We assume that each data type is equipped with a set of *measures* which are *unary* functions whose (1) domain is the data type, and (2) body is a single case-expression over the datatype [Vazou et al. 2014a]:

$$\text{measure } f : \tau = \lambda x. \text{case } y = x \text{ of } \{D_i \bar{z} \rightarrow e_i\}$$

For example, `len` measures the size of an `[Int]`:

```
measure len :: [Int] → Nat
len = \x → case x of
  []      → 0
  (x:xs) → 1 + len xs
```

Checking and Projection We assume the existence of measures that *check* the top-level constructor, and *project* their individual fields. In § F.2 we show how to use these measures to reflect functions over datatypes. For example, for lists, we assume the existence of measures:

```
isNil []      = True
isNil (x:xs) = False

isCons (x:xs) = True
isCons []     = False

sel1 (x:xs) = x
sel2 (x:xs) = xs
```

Refining Data Constructors with Measures We use measures to strengthen the types of data constructors, and we use these strengthened types during construction and destruction (pattern-matching). Let: (1) D be a data constructor, with *unrefined* type $\bar{x} : \bar{\tau} \rightarrow T$ (2) the i -th measure definition with domain T is:

$$\text{measure } f_i : \tau = \lambda x. \text{case } y = x \text{ of } \{D \bar{z} \rightarrow e_i\}$$

Then, the refined type of D is defined:

$$\text{prim}(D) \doteq \bar{x} : \bar{\tau} \rightarrow \{v : T \mid \wedge_i f_i v = e_i [\bar{x}/\bar{z}]\}$$

Thus, each data constructor's output type is refined to reflect the definition of each of its measures. For example, we use the measures `len`, `isNil`, `isCons`, `sel1`, and `sel2` to strengthen the types of `[]` and `:` to:

$$\begin{aligned}\text{prim}([]) &\doteq \{v : [\text{Int}] \mid r_{[]} \} \\ \text{prim}(:) &\doteq x : \text{Int} \rightarrow xs : [\text{Int}] \rightarrow \{v : [\text{Int}] \mid r.\}\end{aligned}$$

where the output refinements are

$$\begin{aligned}r_{[]} &\doteq \text{len } v = 0 \wedge \text{isNil } v \wedge !\text{isCons } v \\ r. &\doteq \text{len } v = 1 + \text{len } xs \wedge !\text{isNil } v \wedge \text{isCons } v \\ &\quad \wedge \text{sel1 } v = x \wedge \text{sel2 } v = xs\end{aligned}$$

It is easy to prove that Lemma D.1 holds for data constructors, by construction. For example, `len [] = 0` evaluates to `true`.

D.6 Typing Rules

Next, we present the type-checking judgments and rules of λ^R .

Environments and Closing Substitutions A *type environment* Γ is a sequence of type bindings $x_1 : \tau_1, \dots, x_n : \tau_n$. An environment denotes a set of *closing substitutions* θ which are sequences of expression bindings: $x_1 \mapsto e_1, \dots, x_n \mapsto e_n$ such that:

$$[[\Gamma]] \doteq \{\theta \mid \forall x : \tau \in \Gamma. \theta(x) \in [[\theta \cdot \tau]]\}$$

Judgments We use environments to define three kinds of rules: Well-formedness, Subtyping, and Typing [Knowles and Flanagan 2010; Vazou et al. 2014a]. A judgment $\Gamma \vdash \tau$ states that the refinement type τ is well-formed in the environment Γ . Intuitively, the type τ is well-formed if all the refinements in τ are `Bool`-typed in Γ . A judgment $\Gamma \vdash \tau_1 \leq \tau_2$ states that the type τ_1 is a subtype of τ_2 in the environment Γ . Informally, τ_1 is a subtype of τ_2 if, when the free variables of τ_1 and τ_2 are bound to some expressions described by Γ , the denotation of τ_1 is *contained in* the denotation of τ_2 . Subtyping of basic types reduces to denotational containment checking. That is, for any closing substitution θ in the denotation of Γ , for every expression e , if $e \in [[\theta \cdot \tau_1]]$ then $e \in [[\theta \cdot \tau_2]]$. A judgment $\Gamma \vdash p : \tau$ states that the program p has the type τ in the environment Γ . That is, when the free variables in p are bound to expressions described by Γ , the program p will evaluate to a value described by τ .

Rules All but three of the rules are standard [Knowles and Flanagan 2010; Vazou et al. 2014a]. First, rule T-REFL is used to strengthen the type of each reflected binder with its definition, as described previously in § D.4. Second, rule T-EXACT strengthens the expression with a singleton type equating the value and the expression (*i.e.* reflecting the expression in the type). This is a generalization of the “selffication” rules from [Knowles and Flanagan 2010; Ou et al. 2004], and is required to equate the reflected functions with their definitions. For example, the application (`fib 1`) is typed as $\{v : \text{Int} \mid \text{fibP } v \ 1 \wedge v = \text{fib } 1\}$ where the first conjunct comes from the (reflection-strengthened) output refinement of `fib` § 2, and the second conjunct comes from rule T-EXACT. Finally, rule T-FIX is used to type the intermediate `fix` expressions that appear, not in the surface language but as intermediate terms in the operational semantics.

Soundness Following λ^U [Vazou et al. 2014a], we can show that evaluation preserves typing and that typing implies denotational inclusion.

THEOREM D.2. [Soundness of λ^R]

- **Denotations** If $\Gamma \vdash p : \tau$ then $\forall \theta \in [[\Gamma]]. \theta \cdot p \in [[\theta \cdot \tau]]$.

Typing

 $\Gamma \vdash p : \tau$

$$\begin{array}{c}
\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \text{ T-VAR} \quad \frac{}{\Gamma \vdash c : \text{prim}(c)} \text{ T-CON} \\
\\
\frac{\Gamma \vdash p : \tau' \quad \Gamma \vdash \tau' \leq \tau}{\Gamma \vdash p : \tau} \text{ T-SUB} \\
\\
\frac{\Gamma \vdash e : \{v : B \mid \{|r|\}\}}{\Gamma \vdash e : \{v : B \mid \{|r|\} \wedge v = e\}} \text{ T-EXACT} \\
\\
\frac{\Gamma, x : \tau_x \vdash e : \tau}{\Gamma \vdash \lambda x. e : x : \tau_x \rightarrow \tau} \text{ T-FUN} \\
\\
\frac{\Gamma \vdash e_1 : (x : \tau_x \rightarrow \tau) \quad \Gamma \vdash e_2 : \tau_x}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-APP} \\
\\
\frac{\Gamma, x : \tau_x \vdash b_x : \tau_x \quad \Gamma, x : \tau_x \vdash \tau_x}{\Gamma, x : \tau_x \vdash b : \tau} \quad \frac{}{\Gamma \vdash \tau} \text{ T-LET} \\
\Gamma \vdash \text{let rec } x : \tau_x = b_x \text{ in } b : \tau \\
\\
\frac{\Gamma \vdash \text{let rec } f : \text{Reflect}(\tau_f, e) = e \text{ in } p : \tau}{\Gamma \vdash \text{reflect } f : \tau_f = e \text{ in } p : \tau} \text{ T-REFL} \\
\\
\frac{\Gamma \vdash e : \{v : T \mid e_r\} \quad \Gamma \vdash \tau}{\forall i. \text{prim}(D_i) = \bar{y}_j : \bar{\tau}_j \rightarrow \{v : T \mid e_{r_i}\}} \quad \frac{\Gamma, \bar{y}_j : \bar{\tau}_j, x : \{v : T \mid e_r \wedge e_{r_i}\} \vdash e_i : \tau}{\Gamma \vdash \text{case } x = e \text{ of } \{D_i \bar{y}_i \rightarrow e_i\} : \tau} \text{ T-CASE}
\end{array}$$

Well Formedness

 $\Gamma \vdash \tau$

$$\begin{array}{c}
\frac{\Gamma, v : B \vdash e : \text{Bool}^{\Downarrow}}{\Gamma \vdash \{v : B \mid e\}} \text{ WF-BASE} \\
\\
\frac{\Gamma \vdash \tau_x \quad \Gamma, x : \tau_x \vdash \tau}{\Gamma \vdash x : \tau_x \rightarrow \tau} \text{ WF-FUN}
\end{array}$$

Subtyping

 $\Gamma \vdash \tau_1 \leq \tau_2$

$$\begin{array}{c}
\frac{\Gamma' \doteq \Gamma, v : \{B^{\Downarrow} \mid e\} \quad \Gamma' \vdash e' \rightsquigarrow p' \quad \text{SmtValid}([\Gamma'] \Rightarrow p')}{\Gamma \vdash \{v : B \mid e\} \leq \{v : B \mid e'\}} \le\text{-BASE-}\lambda^S \\
\\
\frac{\Gamma \vdash \tau'_x \leq \tau_x \quad \Gamma, x : \tau'_x \vdash \tau \leq \tau'}{\Gamma \vdash x : \tau_x \rightarrow \tau \leq x : \tau'_x \rightarrow \tau'} \le\text{-FUN}
\end{array}$$

Fig. 14. Typing of λ^R

- **Preservation** If $\emptyset \vdash p : \tau$ and $p \hookrightarrow^* w$ then $\emptyset \vdash w : \tau$.

D.7 From Programs & Types to Propositions & Proofs

The denotational soundness Theorem D.2 lets us interpret well typed programs as proofs of propositions.

“Definitions” A definition d is a sequence of reflected binders:

$$d ::= \bullet \mid \text{reflect } x : \tau = e \text{ in } d$$

A definition’s environment $\Gamma(d)$ comprises its binders and their reflected types:

$$\begin{aligned} \Gamma(\bullet) &\doteq \emptyset \\ \Gamma(\text{reflect } f : \tau = e \text{ in } d) &\doteq (f, \text{Reflect}(\tau, e)), \Gamma(d) \end{aligned}$$

A definition’s substitution $\theta(d)$ maps each binder to its definition:

$$\begin{aligned} \theta(\bullet) &\doteq [] \\ \theta(\text{reflect } f : \tau = e \text{ in } d) &\doteq [[f/\text{fix } f \ e], \theta(d)] \end{aligned}$$

“Propositions” A proposition is a type

$$x_1 : \tau_1 \rightarrow \dots \rightarrow x_n : \tau_n \rightarrow \{v : \text{Unit} \mid \text{prop}\}$$

For brevity, we abbreviate propositions like the above to $\bar{x} : \bar{\tau} \rightarrow \{\text{prop}\}$ and we call *prop* the proposition’s refinement. For simplicity we assume that $\text{fv}(\tau_i) = \emptyset$.

“Validity”

A proposition $\bar{x} : \bar{\tau} \rightarrow \{\text{prop}\}$ is *valid under* d if

$$\forall \bar{w} \in [[\bar{\tau}]]. \theta(d) \cdot \text{prop} [\bar{w}/\bar{x}] \hookrightarrow^* \text{True}$$

That is, the proposition is valid if its refinement evaluates to True for every (well typed) interpretation for its parameters \bar{x} under d .

“Proofs” A binder b *proves* a proposition τ under d if

$$\emptyset \vdash d[\text{let rec } x : \tau = b \text{ in unit}] : \text{Unit}$$

That is, if the binder b has the proposition’s type τ under the definition d ’s environment.

THEOREM D.3. [Proofs] If b proves τ under d then τ is valid under d .

PROOF. As b proves τ under d , we have

$$\emptyset \vdash d[\text{let rec } x : \tau = b \text{ in unit}] : \text{Unit} \tag{22}$$

By Theorem D.2 on 22 we get

$$\theta(d) \in [[\Gamma(d)]] \tag{23}$$

Furthermore, by the typing rules 22 implies $\Gamma(d) \vdash b : \tau$ and hence, via Theorem D.2

$$\forall \theta \in [[\Gamma(d)]]. \theta \cdot b \in [[\theta \cdot \tau]] \tag{24}$$

Together, 23 and 24 imply

$$\theta(d) \cdot b \in [[\theta(d) \cdot \tau]] \tag{25}$$

By the definition of type denotations, we have

$$[[\theta(d) \cdot \tau]] \doteq \{f \mid \tau \text{ is valid under } d\} \tag{26}$$

By 25, the above set is not empty, and hence τ is valid under d . \square

Example: Fibonacci is increasing In § 2 we verified that under a definition d that includes `fib`, the term `fibUp` proves

$$n : \text{Nat} \rightarrow \{\text{fib } n \leq \text{fib } (n + 1)\}$$

Thus, by Theorem D.3 we get

$$\forall n. 0 \leq n \leftrightarrow^* \text{True} \Rightarrow \text{fib } n \leq \text{fib } (n + 1) \leftrightarrow^* \text{True}$$

E PROOF OF SOUNDNESS

We prove Theorem 4.1 of § D by reduction to Soundness of λ^U [Vazou et al. 2014a].

THEOREM E.1. [Denotations] *If $\Gamma \vdash p : \tau$ then $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot p \in \llbracket \theta \cdot \tau \rrbracket$.*

PROOF. We use the proof from [Vazou et al. 2014b] and specifically Lemma 4 that is identical to the statement we need to prove. Since the proof proceeds by induction in the type derivation, we need to ensure that all the modified rules satisfy the statement.

- **T-EXACT** Assume $\Gamma \vdash e : \{v : B \mid \{r\} \wedge v = e\}$. By inversion $\Gamma \vdash e : \{v : B \mid \{r\}\}$ (1). By (1) and IH we get $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot e \in \llbracket \theta \cdot \{v : B \mid \{r\}\} \rrbracket$. We fix a $\theta \in \llbracket \Gamma \rrbracket$ We get that if $\theta \cdot e \leftrightarrow^* w$, then $\theta \cdot \{r\}[v/w] \leftrightarrow^* \text{True}$. By the Definition of $=$ we get that $w = w \leftrightarrow^* \text{True}$. Since $\theta \cdot (v = e)[v/w] \leftrightarrow^* w = w$, then $\theta \cdot (\{r\} \wedge v = e)[v/w] \leftrightarrow^* \text{True}$. Thus $\theta \cdot e \in \llbracket \theta \cdot \{v : B \mid \{r\} \wedge v = e\} \rrbracket$ and since this holds for any fixed θ , $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot e \in \llbracket \theta \cdot \{v : B \mid \{r\} \wedge v = e\} \rrbracket$.
- **T-LET** Assume $\Gamma \vdash \text{let rec } x : \tau_x = e_x \text{ in } p : \tau$. By inversion $\Gamma, x : \tau_x \vdash e_x : \tau_x$ (1), $\Gamma, x : \tau_x \vdash p : \tau$ (2), and $\Gamma \vdash \tau$ (3). By IH $\forall \theta \in \llbracket \Gamma, x : \tau_x \rrbracket. \theta \cdot e_x \in \llbracket \theta \cdot \tau_x \rrbracket$ (1') $\forall \theta \in \llbracket \Gamma, x : \tau_x \rrbracket. \theta \cdot p \in \llbracket \theta \cdot \tau \rrbracket$ (2'). By (1') and by the type of `fix` $\forall \theta \in \llbracket \Gamma, x : \tau_x \rrbracket. \theta \cdot \text{fix } x e_x \in \llbracket \theta \cdot \tau_x \rrbracket$. By which, (2') and (3) $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot p[\text{fix } x e_x/x] \in \llbracket \theta \cdot \tau \rrbracket$.
- **T-REFL** Assume $\Gamma \vdash \text{reflect } f : \tau_f = e \text{ in } p : \tau$. By inversion, $\Gamma \vdash \text{let rec } f : \text{Reflect}(\tau_f, e) = e \text{ in } p : \tau$. By IH, $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot \text{let rec } f : \text{Reflect}(\tau_f, e) = e \text{ in } p \in \llbracket \theta \cdot \tau \rrbracket$. Since denotations are closed under evaluation, $\forall \theta \in \llbracket \Gamma \rrbracket. \theta \cdot \text{reflect } f : \text{Reflect}(\tau_f, e) = e \text{ in } p \in \llbracket \theta \cdot \tau \rrbracket$.
- **T-FIX** In Theorem 8.3 from [Vazou et al. 2014b] (and using the textbook proofs from [Rocca and Paolini 2004]) we proved that for each type τ , $\text{fix } \tau \in \llbracket (\tau \rightarrow \tau) \rightarrow \tau \rrbracket$.

\square

THEOREM E.2. [Preservation] *If $\emptyset \vdash p : \tau$ and $p \leftrightarrow^* w$ then $\emptyset \vdash w : \tau$.*

PROOF. In [Vazou et al. 2014b] proof proceeds by iterative application of Type Preservation Lemma 7. Thus, it suffices to ensure Type Preservation in λ^R , which it true by the following Lemma. \square

LEMMA E.3. *If $\emptyset \vdash p : \tau$ and $p \hookrightarrow p'$ then $\emptyset \vdash p' : \tau$.*

PROOF. Since Type Preservation in λ^U is proved by induction on the type derivation tree, we need to ensure that all the modified rules satisfy the statement.

- **T-EXACT** Assume $\emptyset \vdash p : \{v : B \mid \{r\} \wedge v = p\}$. By inversion $\emptyset \vdash p : \{v : B \mid \{r\}\}$. By IH we get $\emptyset \vdash p' : \{v : B \mid \{r\}\}$. By rule T-EXACT we get $\emptyset \vdash p' : \{v : B \mid \{r\} \wedge v = p'\}$. Since subtyping is closed under evaluation, we get $\emptyset \vdash \{v : B \mid \{r\} \wedge v = p'\} \leq \{v : B \mid \{r\} \wedge v = p\}$. By rule T-SUB we get $\emptyset \vdash p' : \{v : B \mid \{r\} \wedge v = p\}$.

Predicates	p	$::=$	$p \bowtie p \mid \oplus_1 p$ $\mid n \mid b \mid x \mid D \mid x \bar{p}$ $\mid \text{if } p \text{ then } p \text{ else } p$
Integers	n	$::=$	$0, -1, 1, \dots$
Booleans	b	$::=$	$\text{True} \mid \text{False}$
Bin Operators	\bowtie	$::=$	$= \mid < \mid \wedge \mid + \mid - \mid \dots$
Un Operators	\oplus_1	$::=$	$! \mid \dots$
Model	σ	$::=$	$\sigma, (x : p) \mid \emptyset$
Sort Arguments	s_a	$::=$	$\text{Int} \mid \text{Bool} \mid \text{U} \mid \text{Fun } s_a s_a$
Sorts	s	$::=$	$s_a \rightarrow s$

Fig. 15. Syntax of λ^S

- T-LET Assume $\emptyset \vdash \text{let rec } x : \tau_x = e_x \text{ in } p : \tau$. By inversion, $x : \tau_x \vdash e_x : \tau_x$ (1), $x : \tau_x \vdash p : \tau$ (2), and $\Gamma \vdash \tau$ (3). By rule T-FIX $x : \tau_x \vdash \text{fix } x e_x : \tau_x$ (1'). By (1'), (2) and Lemma 6 of [Vazou et al. 2014b], we get $\vdash p [\text{fix } x e_x/x] : \tau [\text{fix } x e_x/x]$. By (3) $\tau [\text{fix } x e_x/x] \equiv \tau$. Since $p' \equiv p [\text{fix } x e_x/x]$, we have $\emptyset \vdash p' : \tau$.
- T-REFL Assume $\emptyset \vdash \text{reflect } x : \tau_x = e_x \text{ in } p : \tau$. By double inversion, with $\tau'_x \equiv \text{Reflect}(\tau_x, e_x)$; $x : \tau'_x \vdash e_x : \tau'_x$ (1), $x : \tau'_x \vdash p : \tau$ (2), and $\Gamma \vdash \tau$ (3). By rule T-FIX $x : \tau'_x \vdash \text{fix } x e_x : \tau'_x$ (1'). By (1'), (2) and Lemma 6 of [Vazou et al. 2014b], we get $\vdash p [\text{fix } x e_x/x] : \tau [\text{fix } x e_x/x]$. By (3) $\tau [\text{fix } x e_x/x] \equiv \tau$. Since $p' \equiv p [\text{fix } x e_x/x]$, we have $\emptyset \vdash p' : \tau$.
- T-FIX This case cannot occur, as `fix` does not evaluate to any program.

□

F ALGORITHMIC CHECKING λ^S : EXTENDED VERSION

Next, we describe λ^S , a conservative approximation of λ^R where the undecidable type subsumption rule is replaced with a decidable one, yielding an SMT-based algorithmic type system that enjoys the same soundness guarantees.

F.1 The SMT logic λ^S

Syntax: Terms & Sorts Figure 15 summarizes the syntax of λ^S , the *sorted* (SMT-) decidable logic of quantifier-free equality, uninterpreted functions and linear arithmetic (QF-EUFLIA) [Barrett et al. 2010; Nelson 1980]. The *terms* of λ^S include integers n , booleans b , variables x , data constructors D (encoded as constants), fully applied unary \oplus_1 and binary \bowtie operators, and application $x \bar{p}$ of an uninterpreted function x . The *sorts* of λ^S include built-in integer `Int` and `Bool` for representing integers and booleans. The interpreted functions of λ^S , e.g. the logical constants `=` and `<`, have the function sort $s \rightarrow s$. Other functional values in λ^R , e.g. reflected λ^R functions and λ -expressions, are represented as first-order values with uninterpreted sort `Fun` $s s$. The universal sort `U` represents all other values.

Semantics: Satisfaction & Validity An assignment σ is a mapping from variables to terms $\sigma \doteq \{x_1 \mapsto p_1, \dots, x_n \mapsto p_n\}$. We write $\sigma \models p$ if the assignment σ is a *model* of p , intuitively if σp “is true” [Nelson 1980]. A predicate p is *satisfiable* if there exists $\sigma \models p$. A predicate p is *valid* if for all assignments $\sigma \models p$.

Transformation

$$\boxed{\Gamma \vdash e \rightsquigarrow p}$$

$$\frac{}{\Gamma \vdash b \rightsquigarrow b} \quad - \models B \hookrightarrow^* \text{ool} \quad \frac{}{\Gamma \vdash n \rightsquigarrow n} \quad - \models I \hookrightarrow^* \text{nt}$$

$$\frac{\Gamma \vdash e_1 \rightsquigarrow p_1 \quad \Gamma \vdash e_2 \rightsquigarrow p_2}{\Gamma \vdash e_1 \bowtie e_2 \rightsquigarrow p_1 \bowtie p_2} \quad - \models B \hookrightarrow^* \text{in}$$

$$\frac{\Gamma \vdash e \rightsquigarrow p}{\Gamma \vdash \oplus_1 e \rightsquigarrow \oplus_1 p} \quad - \models U \hookrightarrow^* n \quad \frac{}{\Gamma \vdash x \rightsquigarrow x} \quad - \models V \hookrightarrow^* \text{ar}$$

$$\frac{}{\Gamma \vdash c \rightsquigarrow s_c} \quad - \models O \hookrightarrow^* p \quad \frac{}{\Gamma \vdash D \rightsquigarrow s_D} \quad - \models D \hookrightarrow^* C$$

$$\frac{\Gamma, x : \tau_x \vdash e \rightsquigarrow p \quad \Gamma \vdash (\lambda x. e) : (x : \tau_x \rightarrow \tau)}{\Gamma \vdash \lambda x. e \rightsquigarrow \text{lam}_{\binom{\tau_x}{\tau}} x p} \quad - \models F \hookrightarrow^* \text{un}$$

$$\frac{\Gamma \vdash e' \rightsquigarrow p' \quad \Gamma \vdash e \rightsquigarrow p \quad \Gamma \vdash e : \tau_x \rightarrow \tau}{\Gamma \vdash e e' \rightsquigarrow \text{app}_{\binom{\tau_x}{\tau}} p p'} \quad - \models A \hookrightarrow^* \text{pp}$$

$$\frac{\Gamma \vdash e \rightsquigarrow p \quad \Gamma \vdash e_i[x/e] \rightsquigarrow p_i}{\Gamma \vdash \text{case } x = e \text{ of } \{\text{True} \rightarrow e_1; \text{False} \rightarrow e_2\} \rightsquigarrow \text{if } p \text{ then } p_1 \text{ else } p_2} \quad - \models I \hookrightarrow^* f$$

$$\frac{\Gamma \vdash e \rightsquigarrow p}{\Gamma \vdash e_i[\overline{y_i}/\text{sel}_{D_i} x][x/e] \rightsquigarrow p_i} \quad - \models C \hookrightarrow^* \text{ase}$$

$$\frac{}{\Gamma \vdash \text{case } x = e \text{ of } \{D_i \overline{y_i} \rightarrow e_i\} \rightsquigarrow \text{if app is}_{D_i} p \text{ then } p_1 \text{ else } \dots \text{ else } p_n}$$

Fig. 16. Transforming λ^R terms into λ^S .**F.2 Transforming λ^R into λ^S**

The judgment $\Gamma \vdash e \rightsquigarrow p$ states that a λ^R term e is transformed, under an environment Γ , into a λ^S term p . The transformation rules are summarized in Figure 16.

Embedding Types We embed λ^R types into λ^S sorts as:

$$\begin{aligned} \langle \text{Int} \rangle &\doteq \text{Int} & \langle T \rangle &\doteq \text{U} \\ \langle \text{Bool} \rangle &\doteq \text{Bool} & \langle x : \tau_x \rightarrow \tau \rangle &\doteq \text{Fun } \langle \tau_x \rangle \langle \tau \rangle \end{aligned}$$

Embedding Constants Elements shared on both λ^R and λ^S translate to themselves. These elements include booleans ($- \models B \hookrightarrow^* \text{ool}$), integers ($- \models I \hookrightarrow^* \text{nt}$), variables ($- \models V \hookrightarrow^* \text{ar}$), binary ($- \models B \hookrightarrow^* \text{in}$) and unary ($- \models U \hookrightarrow^* n$) operators. SMT solvers do not support currying, and so in λ^S , all function symbols must be fully applied. Thus, we assume that all applications to primitive constants and data constructors are *saturated*, i.e. fully applied, e.g. by converting source level terms like $(+ 1)$ to $(\lambda z \rightarrow z + 1)$.

Embedding Functions As λ^S is a first-order logic, we embed λ -abstraction and application using the uninterpreted functions `lam` and `app`. We embed λ -abstractions using `lam` as shown in

rule $- \models F \hookrightarrow^* uN$. The term $\lambda x.e$ of type $\tau_x \rightarrow \tau$ is transformed to $\text{lam}^{s_x} x p$ of sort $\text{Fun } s_x s$, where s_x and s are respectively (τ_x) and (τ) , lam^{s_x} is a special uninterpreted function of sort $s_x \rightarrow s \rightarrow \text{Fun } s_x s$, and x of sort s_x and r of sort s are the embedding of the binder and body, respectively. As lam is just an SMT-function, it *does not* create a binding for x . Instead, the binder x is renamed to a *fresh* name pre-declared in the SMT environment.

Embedding Applications Dually, we embed applications via defunctionalization [Reynolds 1972] using an uninterpreted *apply* function app as shown in rule $- \models A \hookrightarrow^* pP$. The term $e e'$, where e and e' have types $\tau_x \rightarrow \tau$ and τ_x , is transformed to $\text{app}_s^{s_x} p p' : s$ where s and s_x are respectively (τ) and (τ_x) , the $\text{app}_s^{s_x}$ is a special uninterpreted function of sort $\text{Fun } s_x s \rightarrow s_x \rightarrow s$, and p and p' are the respective translations of e and e' .

Embedding Data Types Rule $- \models D \hookrightarrow^* C$ translates each data constructor to a predefined λ^S constant s_D of sort $(\text{prim}(D))$. Let D_i be a non-boolean data constructor such that

$$\text{prim}(D_i) \doteq \tau_{i,1} \rightarrow \dots \rightarrow \tau_{i,n} \rightarrow \tau$$

Then the *check function* is_{D_i} has the sort $\text{Fun } (\tau) \text{Bool}$, and the *select function* $\text{sel}_{D_i,j}$ has the sort $\text{Fun } (\tau) (\tau_{i,j})$. Rule $- \models C \hookrightarrow^* aSE$ translates case-expressions of λ^R into nested *if* terms in λ^S , by using the check functions in the guards, and the select functions for the binders of each case. For example, following the above, the body of the list append function

```
[ ]      ++ ys = ys
(x : xs) ++ ys = x : (xs ++ ys)
```

is reflected into the λ^S refinement:

```
if isNil xs then ys else sel1 xs : (sel2 xs ++ ys)
```

We favor selectors to the axiomatic translation of HALO [Vytiniotis et al. 2013] and F* [Swamy et al. 2016] to avoid universally quantified formulas and the resulting instantiation unpredictability.

F.3 Correctness of Translation

Informally, the translation relation $\Gamma \vdash e \rightsquigarrow p$ is correct in the sense that if e is a terminating boolean expression then e reduces to *True* iff p is SMT-satisfiable by a model that respects β -equivalence.

Definition F.1 (β -Model). A β -model σ^β is an extension of a model σ where lam and app satisfy the axioms of β -equivalence:

$$\begin{aligned} \forall x y e. \text{lam } x e &= \text{lam } y (e[x/y]) \\ \forall x e_x e. (\text{app } (\text{lam } x e) e_x &= e[x/e_x]) \end{aligned}$$

Semantics Preservation We define the translation of a λ^R term into λ^S under the empty environment as $(e) \doteq p$ if $\emptyset \vdash e \rightsquigarrow p$. A *lifted substitution* θ^\perp is a set of models σ where each “bottom” in the substitution θ is mapped to an arbitrary logical value of the respective sort [Vazou et al. 2014a]. We connect the semantics of λ^R and translated λ^S via the following theorems:

THEOREM F.2. *If $\Gamma \vdash e \rightsquigarrow p$, then for every $\theta \in \llbracket \Gamma \rrbracket$ and every $\sigma \in \theta^\perp$, if $\theta^\perp \cdot e \hookrightarrow^* v$ then $\sigma^\beta \models p = (v)$.*

COROLLARY F.3. *If $\Gamma \vdash e : \text{Bool}$, e reduces to a value and $\Gamma \vdash e \rightsquigarrow p$, then for every $\theta \in \llbracket \Gamma \rrbracket$ and every $\sigma \in \theta^\perp$, $\theta^\perp \cdot e \hookrightarrow^* \text{True}$ iff $\sigma^\beta \models p$.*

Refined Types $\tau ::= \{v : B^{\Downarrow} \mid \{ \} \mid \} x : \tau \rightarrow \tau$

Well Formedness

$\Gamma \vdash_S \tau$

$$\frac{\Gamma, v : B \vdash_S e : \text{Bool}^{\Downarrow}}{\Gamma \vdash_S \{v : B \mid e\}} \text{WF-BASE}$$

Subtyping

$\Gamma \vdash_S \tau \leq \tau'$

$$\frac{\Gamma' \doteq \Gamma, v : \{B^{\Downarrow} \mid e\} \quad \Gamma' \vdash e' \rightsquigarrow p' \quad \text{SmtValid}(\lfloor \Gamma' \rceil \Rightarrow p')}{\Gamma \vdash_S \{v : B \mid e\} \leq \{v : B \mid e'\}} \leq\text{-BASE-}\lambda^S$$

Fig. 17. **Algorithmic Typing (other rules in Figs 12 and 14.)**

F.4 Decidable Type Checking

Figure 17 summarizes the modifications required to obtain decidable type checking. Namely, basic types are extended with labels that track termination and subtyping is checked via an SMT solver.

Termination Under arbitrary beta-reduction semantics (which includes lazy evaluation), soundness of refinement type checking requires checking termination, for two reasons: (1) to ensure that refinements cannot diverge, and (2) to account for the environment during subtyping [Vazou et al. 2014a]. We use \Downarrow to mark provably terminating computations, and extend the rules to use refinements to ensure that if $\Gamma \vdash_S e : \{v : B^{\Downarrow} \mid r\}$, then e terminates [Vazou et al. 2014a].

Verification Conditions The *verification condition* (VC) $\lfloor \Gamma \rceil \Rightarrow p$ is *valid* only if the set of values described by Γ , is subsumed by the set of values described by p . Γ is embedded into logic by conjoining (the embeddings of) the refinements of provably terminating binders [Vazou et al. 2014a]:

$$\lfloor \Gamma \rceil \doteq \bigwedge_{x \in \Gamma} \lfloor \Gamma, x \rceil$$

where we embed each binder as

$$\lfloor \Gamma, x \rceil \doteq \begin{cases} p & \text{if } \Gamma(x) = \{v : B^{\Downarrow} \mid e\}, \Gamma \vdash e[v/x] \rightsquigarrow p \\ \text{True} & \text{otherwise.} \end{cases}$$

Subtyping via SMT Validity We make subtyping, and hence, typing decidable, by replacing the denotational base subtyping rule $\leq\text{-BASE-}\lambda^S$ with a conservative, algorithmic version that uses an SMT solver to check the validity of the subtyping VC. We use Corollary F.3 to prove soundness of subtyping.

LEMMA F.4. *If $\Gamma \vdash_S \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$ then $\Gamma \vdash \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$.*

Soundness of λ^S Lemma F.4 directly implies the soundness of λ^S .

THEOREM F.5 (SOUNDNESS OF λ^S). *If $\Gamma \vdash_S e : \tau$ then $\Gamma \vdash e : \tau$.*

G SOUNDNESS OF ALGORITHMIC VERIFICATION

In this section we prove soundness of Algorithmic verification, by proving the theorems of § 5 by referring to the proofs in [Vazou et al. 2014b].

G.1 Transformation

Definition G.1 (Initial Environment). We define the initial SMT environment Δ_0 to include

s_c	: $\langle \text{prim}(c) \rangle$	$\forall c \in \lambda^R$
$\text{lam}_s^{s_x}$: $s_x \rightarrow s \rightarrow \text{Fun } s_x \ s$	$\forall s_x, s \in \lambda^S$
$\text{app}_s^{s_x}$: $\text{Fun } s_x \ s \rightarrow s_x \rightarrow s$	$\forall s_x, s \in \lambda^S$
s_D	: $\langle \text{prim}(D) \rangle$	$\forall D \in \lambda^R$
$i s_D$: $\langle T \rightarrow \text{Bool} \rangle$	$\forall D \in \lambda^R$ of data type T
sel_{D_i}	: $\langle T \rightarrow \tau_i \rangle$	$\forall D \in \lambda^R$ of data type T and i -th argument τ_i
x_i^s	: s	$\forall s \in \lambda^S$ and $1 \leq i \leq M_\lambda$

Where x_i^s are M_λ global names that only appear as lambda arguments.

We modify the $- \models F \hookrightarrow^* uN$ rule to ensure that logical abstraction is performed using the minimum globally defined lambda argument that is not already abstracted. We do so, using the helper function $\text{MaxLam}(s, p)$:

$$\begin{aligned}
 \text{MaxLam}(s, \text{lam}_s^s, x_i^s \ p) &= \max(i, \text{MaxLam}(s, p)) \\
 \text{MaxLam}(s, r \ \bar{r}) &= \max(\text{MaxLam}(s, p, \bar{p})) \\
 \text{MaxLam}(s, p_1 \bowtie p_2) &= \max(\text{MaxLam}(s, p_1, p_2)) \\
 \text{MaxLam}(s, \oplus_1 p) &= \text{MaxLam}(s, p) \\
 \text{MaxLam}(s, \text{if } p \text{ then } p_1 \text{ else } p_2) &= \max(\text{MaxLam}(s, p, p_1, p_2)) \\
 \text{MaxLam}(s, p) &= 0
 \end{aligned}$$

$$\frac{
 \begin{array}{l}
 i = \text{MaxLam}(\langle \tau_x \rangle, p) \quad i < M_\lambda \quad y = x_{i+1}^{\langle \tau_x \rangle} \\
 \Gamma, y : \tau_x \vdash e[x/y] \rightsquigarrow p \quad \Gamma \vdash (\lambda x. e) : (x : \tau_x \rightarrow \tau)
 \end{array}
 }{
 \Gamma \vdash \lambda x. e \rightsquigarrow \text{lam}_{\langle \tau \rangle}^{\langle \tau_x \rangle} y \ p
 } \quad - \models F \hookrightarrow^* uN$$

LEMMA G.2 (TYPE TRANSFORMATION). *If $\Gamma \vdash e \rightsquigarrow p$, and $\Gamma \vdash e : \tau$, then $\Delta_0, \langle \Gamma \rangle \vdash_S p : \langle \tau \rangle$.*

PROOF. We proceed by induction on the translation

- $- \models B \hookrightarrow^* \text{ool}$: Since $\langle \text{Bool} \rangle = \text{Bool}$, If $\Gamma \vdash b : \text{Bool}$, then $\Delta_0, \langle \Gamma \rangle \vdash_S b : \langle \text{Bool} \rangle$.
- $- \models I \hookrightarrow^* nT$: Since $\langle \text{Int} \rangle = \text{Int}$, If $\Gamma \vdash n : \text{Int}$, then $\Delta_0, \langle \Gamma \rangle \vdash_S n : \langle \text{Int} \rangle$.
- $- \models U \hookrightarrow^* n$: Since $\Gamma \vdash ! e : \tau$, then it should be $\Gamma \vdash e : \text{Bool}$ and $\tau \equiv \text{Bool}$. By IH, $\Delta_0, \langle \Gamma \rangle \vdash_S p : \langle \text{Bool} \rangle$, thus $\Delta_0, \langle \Gamma \rangle \vdash_S !p : \langle \text{Bool} \rangle$.
- $- \models B \hookrightarrow^* iN$ Assume $\Gamma \vdash e_1 \bowtie e_2 \rightsquigarrow p_1 \bowtie p_2$. By inversion $\Gamma \vdash e_1 \rightsquigarrow p_1$, and $\Gamma \vdash e_2 \rightsquigarrow p_2$. Since $\Gamma \vdash e_1 \bowtie e_2 : \tau$, then $\Gamma \vdash e_1 : \tau_1$ and $\Gamma \vdash e_2 : \tau_2$. By IH, $\Delta_0, \langle \Gamma \rangle \vdash_S p_1 : \langle \tau_1 \rangle$ and $\Delta_0, \langle \Gamma \rangle \vdash_S p_2 : \langle \tau_2 \rangle$. We split cases on \bowtie
 - If $\bowtie \equiv =$, then $\tau_1 = \tau_2$, thus $\langle \tau_1 \rangle = \langle \tau_2 \rangle$ and $\langle \tau \rangle = \tau = \text{Bool}$.
 - If $\bowtie \equiv <$, then $\tau_1 = \tau_2 = \text{Int}$, thus $\langle \tau_1 \rangle = \langle \tau_2 \rangle = \text{Int}$ and $\langle \tau \rangle = \tau = \text{Bool}$.
 - If $\bowtie \equiv \wedge$, then $\tau_1 = \tau_2 = \text{Bool}$, thus $\langle \tau_1 \rangle = \langle \tau_2 \rangle = \text{Bool}$ and $\langle \tau \rangle = \tau = \text{Bool}$.
 - If $\bowtie \equiv +$ or $\bowtie \equiv -$, then $\tau_1 = \tau_2 = \text{Int}$, thus $\langle \tau_1 \rangle = \langle \tau_2 \rangle = \text{Int}$ and $\langle \tau \rangle = \tau = \text{Int}$.
- $- \models V \hookrightarrow^* aR$: Assume $\Gamma \vdash x \rightsquigarrow x$ Then $\Gamma \vdash x : \Gamma(x)$ and $\Delta_0, \langle \Gamma \rangle \vdash_S x : \langle \Gamma \rangle(x)$. But by definition $\langle \langle \Gamma \rangle \rangle(x) = \langle \Gamma(x) \rangle$.
- $- \models O \hookrightarrow^* p$: Assume $\Gamma \vdash c \rightsquigarrow s_c$ Also, $\Gamma \vdash c : \text{prim}(c)$ and $\Delta_0, \langle \Gamma \rangle \vdash_S s_c : \Delta_0(s_c)$. But by Definition G.1 $\Delta_0(s_c) = \langle \text{prim}(c) \rangle$.

- $- \models D \hookrightarrow^* C$: Assume $\Gamma \vdash D \rightsquigarrow s_D$. Also, $\Gamma \vdash D : \text{prim}(D)$ and $\Delta_0, (\Gamma) \vdash_S s_D : \Delta_0(s_D)$. But by Definition G.1 $\Delta_0(s_D) = (\text{prim}(c))$.
- $- \models F \hookrightarrow^* uN$: Assume $\Gamma \vdash \lambda x.e \rightsquigarrow \text{lam}_{(\tau)}^{(\tau_x)} x_i^{(\tau_x)} p$. By inversion $i \leq M_\lambda$, $\Gamma, x_i^{(\tau_x)} : \tau_x \vdash e[x/x_i^{(\tau_x)}] \rightsquigarrow p$, and $\Gamma \vdash (\lambda x.e) : (x : \tau_x \rightarrow \tau)$. By the Definition G.1 on lam , $x_i^{(\tau_x)}$ and induction, we get $\Delta_0, (\Gamma) \vdash_S \text{lam}_{(\tau)}^{(\tau_x)} x_i^{(\tau_x)} p : \text{Fun}(\tau_x)(\tau)$. By the definition of the type embeddings we have $(x \rightarrow \tau_x \tau) \doteq \text{Fun}(\tau_x)(\tau)$.
- $- \models A \hookrightarrow^* pP$: Assume $\Gamma \vdash e e' \rightsquigarrow \text{app}_{(\tau)}^{(\tau_x)} p p'$. By inversion, $\Gamma \vdash e' \rightsquigarrow p'$, $\Gamma \vdash e \rightsquigarrow p$, $\Gamma \vdash e : x : \tau_x \rightarrow \tau$. By IH and the type of app we get that $\Delta_0, (\Gamma) \vdash_S \text{app}_{(\tau)}^{(\tau_x)} p p' : (\tau)$.
- $- \models I \hookrightarrow^* f$: Assume $\Gamma \vdash \text{case } x = e \text{ of } \{\text{True} \rightarrow e_1; \text{False} \rightarrow e_2\} \rightsquigarrow \text{if } p \text{ then } p_1 \text{ else } p_2$. Since $\Gamma \vdash \text{case } x = e \text{ of } \{\text{True} \rightarrow e_1; \text{False} \rightarrow e_2\} : \tau$, then $\Gamma \vdash e : \text{Bool}$, $\Gamma \vdash e_1 : \tau$, and $\Gamma \vdash e_2 : \tau$. By inversion and IH, $\Delta_0, (\Gamma) \vdash_S p : \text{Bool}$, $\Delta_0, (\Gamma) \vdash_S p_1 : (\tau)$, and $\Delta_0, (\Gamma) \vdash_S p_2 : (\tau)$. Thus, $\Delta_0, (\Gamma) \vdash_S \text{if } p \text{ then } p_1 \text{ else } p_2 : (\tau)$.
- $- \models C \hookrightarrow^* aS$: Assume $\Gamma \vdash \text{case } x = e \text{ of } \{D_i \bar{y}_i \rightarrow e_i\} \rightsquigarrow \text{if is}_{D_1} p \text{ then } p_1 \text{ else } \dots \text{ else } p_n$ and $\Gamma \vdash \text{case } x = e \text{ of } \{D_i \bar{y}_i \rightarrow e_i\} : \tau$. By inversion we get $\Gamma \vdash e \rightsquigarrow p$ and $\Gamma \vdash e_i[\bar{y}_i/\text{sel}_{D_i} x][x/e] \rightsquigarrow p_i$. By IH and the Definition G.1 on the checkers and selectors, we get $\Delta_0, (\Gamma) \vdash_S \text{if is}_{D_1} p \text{ then } p_1 \text{ else } \dots \text{ else } p_n : (\tau)$.

□

THEOREM G.3. *If $\Gamma \vdash e \rightsquigarrow p$, then for every substitution $\theta \in \llbracket \Gamma \rrbracket$ and every model $\sigma \in \llbracket \theta^\perp \rrbracket$, if $\theta^\perp \cdot e \hookrightarrow^* v$ then $\sigma^\beta \models p = \llbracket v \rrbracket$.*

PROOF. We proceed using the notion of tracking substitutions from Figure 8 of [Vazou et al. 2014b]. Since $\theta^\perp \cdot e \hookrightarrow^* v$, there exists a sequence of evaluations via tracked substitutions,

$$\langle \theta_1^\perp; e_1 \rangle \hookrightarrow \dots \langle \theta_i^\perp; e_i \rangle \dots \hookrightarrow \langle \theta_n^\perp; e_n \rangle$$

with $\theta_1^\perp \equiv \theta^\perp$, $e_1 \equiv e$, and $e_n \equiv v$. Moreover, each e_{i+1} is well formed under Γ , thus it has a translation $\Gamma \vdash e_{i+1} \rightsquigarrow p_{i+1}$. Thus we can iteratively apply Lemma G.5 $n - 1$ times and since v is a value the extra variables in θ_n^\perp are irrelevant, thus we get the required $\sigma^\beta \models p = \llbracket v \rrbracket$. □

For Boolean expressions we specialize the above to

COROLLARY G.4. *If $\Gamma \vdash e : \text{Bool}^\perp$ and $\Gamma \vdash e \rightsquigarrow p$, then for every substitution $\theta \in \llbracket \Gamma \rrbracket$ and every model $\sigma \in \llbracket \theta^\perp \rrbracket$, $\theta^\perp \cdot e \hookrightarrow^* \text{True} \iff \sigma^\beta \models p$*

PROOF. We prove the left and right implication separately:

- \implies By direct application of Theorem G.3 for $v \equiv \text{True}$.
- \impliedby Since e is terminating, $\theta^\perp \cdot e \hookrightarrow^* v$. with either $v \equiv \text{True}$ or $v \equiv \text{False}$. Assume $v \equiv \text{False}$, then by Theorem G.3, $\sigma^\beta \models !p$, which is a contradiction. Thus, $v \equiv \text{True}$.

□

LEMMA G.5 (EQUIVALENCE PRESERVATION). *If $\Gamma \vdash e \rightsquigarrow p$, then for every substitution $\theta \in \llbracket \Gamma \rrbracket$ and every model $\sigma \in \llbracket \theta^\perp \rrbracket$, if $\langle \theta^\perp; e \rangle \hookrightarrow \langle \theta_2^\perp; e_2 \rangle$ and for $\Gamma \subseteq \Gamma_2$ so that $\theta_2^\perp \in \llbracket \Gamma_2 \rrbracket$ and $\sigma_2^\beta \in \llbracket \theta_2^\perp \rrbracket$, $\Gamma_2 \vdash e_2 \rightsquigarrow p_2$ then $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models p = p_2$.*

PROOF. We proceed by case analysis on the derivation $\langle \theta^\perp; e \rangle \hookrightarrow \langle \theta_2^\perp; e_2 \rangle$.

- Assume $\langle \theta^\perp; e_1 e_2 \rangle \hookrightarrow \langle \theta_2^\perp; e'_1 e_2 \rangle$. By inversion $\langle \theta^\perp; e_1 \rangle \hookrightarrow \langle \theta_2^\perp; e'_1 \rangle$. Assume $\Gamma \vdash e_1 \rightsquigarrow p_1$, $\Gamma \vdash e_2 \rightsquigarrow p_2$, $\Gamma_2 \vdash e'_1 \rightsquigarrow p'_1$. By IH $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models p_1 = p'_1$, thus $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models \text{app } p_1 p_2 = \text{app } p'_1 p_2$.

- Assume $\langle \theta^\perp; c e \rangle \hookrightarrow \langle \theta_2^\perp; c e' \rangle$. By inversion $\langle \theta^\perp; e \rangle \hookrightarrow \langle \theta_2^\perp; e' \rangle$. Assume $\Gamma \vdash e \rightsquigarrow p$, $\Gamma \vdash e' \rightsquigarrow p'$. By IH $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models p = p'$, thus $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models \text{app } c p = \text{app } c p'$.
- Assume $\langle \theta^\perp; \text{case } x = e \text{ of } \{D_i \bar{y}_i \rightarrow e_i\} \rangle \hookrightarrow \langle \theta_2^\perp; \text{case } x = e' \text{ of } \{D_i \bar{y}_i \rightarrow e_i\} \rangle$. By inversion $\langle \theta^\perp; e \rangle \hookrightarrow \langle \theta_2^\perp; e' \rangle$. Assume $\Gamma \vdash e \rightsquigarrow p$, $\Gamma \vdash e' \rightsquigarrow p'$. $\Gamma \vdash e_i[\bar{y}_i/\overline{\text{se}}_{D_i} x][x/e] \rightsquigarrow p_i$. By IH $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models p = p'$, thus $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models \text{if is}_{D_i} p \text{ then } p_1 \text{ else } \dots \text{ else } p_n(\tau) = \text{if is}_{D_i} p' \text{ then } p_1 \text{ else } \dots \text{ else } p_n(\tau)$.
- Assume $\langle \theta^\perp; D \bar{e}_i e' \bar{e}_j \rangle \hookrightarrow \langle \theta_2^\perp; D \bar{e}_i e' \bar{e}_j \rangle$. By inversion $\langle \theta^\perp; e \rangle \hookrightarrow \langle \theta_2^\perp; e' \rangle$. Assume $\Gamma \vdash e \rightsquigarrow p$, $\Gamma \vdash e_i \rightsquigarrow p_i, \Gamma \vdash e' \rightsquigarrow p'$. By IH $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models p = p'$, thus $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models \text{app } D \bar{p}_i p \bar{p}_j = \text{app } D \bar{p}_i p' \bar{p}_j$.
- Assume $\langle \theta^\perp; c w \rangle \hookrightarrow \langle \theta^\perp; \delta(c, w) \rangle$. By the definition of the syntax, $c w$ is a fully applied logical operator, thus $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models c w = \llbracket \delta(c, w) \rrbracket$
- Assume $\langle \theta^\perp; (\lambda x. e) e_x \rangle \hookrightarrow \langle \theta^\perp; e[x/e_x] \rangle$. Assume $\Gamma \vdash e \rightsquigarrow p, \Gamma \vdash e_x \rightsquigarrow p_x$. Since σ^β is defined to satisfy the β -reduction axiom, $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models \text{app } (\lambda x e) p_x = p[x/p_x]$.
- Assume $\langle \theta^\perp; \text{case } x = D_j \bar{e} \text{ of } \{D_i \bar{y}_i \rightarrow e_i\} \rangle \hookrightarrow \langle \theta^\perp; e_j[x/D_j \bar{e}][y_i/\bar{e}_i] \rangle$. Also, let $\Gamma \vdash e \rightsquigarrow p$, $\Gamma \vdash e_i[x/D_j \bar{e}][y_i/\bar{e}_i] \rightsquigarrow p_i$. By the axiomatic behavior of the measure selector $\text{is}_{D_j} \bar{p}$, we get $\sigma^\beta \models \text{is}_{D_j} \bar{p}$. Thus, $\sigma^\beta \models \text{if is}_{D_j} p \text{ then } p_1 \text{ else } \dots \text{ else } p_n = p_j$.
- Assume $\langle (x, e_x) \theta^\perp; x \rangle \hookrightarrow \langle (x, e'_x) \theta_2^\perp; x \rangle$. By inversion $\langle \theta^\perp; e_x \rangle \hookrightarrow \langle \theta_2^\perp; e'_x \rangle$. By identity of equality, $(x, p_x) \sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models x = x$.
- Assume $\langle (y, e_y) \theta^\perp; x \rangle \hookrightarrow \langle (y, e_y) \theta_2^\perp; e_x \rangle$. By inversion $\langle \theta^\perp; x \rangle \hookrightarrow \langle \theta_2^\perp; e_x \rangle$. Assume $\Gamma \vdash e_x \rightsquigarrow p_x$. By IH $\sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models x = p_x$. Thus $(y, p_y) \sigma^\beta \cup (\sigma_2^\beta \setminus \sigma^\beta) \models x = p_x$.
- Assume $\langle (x, w) \theta^\perp; x \rangle \hookrightarrow \langle (x, w) \theta^\perp; w \rangle$. Thus $(x, \llbracket w \rrbracket) \sigma^\beta \models x = \llbracket w \rrbracket$.
- Assume $\langle (x, D \bar{y}) \theta^\perp; x \rangle \hookrightarrow \langle (x, D \bar{y}) \theta^\perp; D \bar{y} \rangle$. Thus $(x, \text{app } D \bar{y}) \sigma^\beta \models x = \text{app } D \bar{y}$.
- Assume $\langle (x, D \bar{e}) \theta^\perp; x \rangle \hookrightarrow \langle (x, D \bar{y}), \overline{(y_i, e_i)} \theta^\perp; D \bar{y} \rangle$. Assume $\Gamma \vdash e_i \rightsquigarrow p_i$. Thus $(x, \text{app } D \bar{y}), \overline{(y_i, p_i)} \sigma^\beta \models x = \text{app } D \bar{y}$.

□

G.2 Soundness of Approximation

THEOREM G.6 (SOUNDNESS OF ALGORITHMIC). *If $\Gamma \vdash_S e : \tau$ then $\Gamma \vdash e : \tau$.*

PROOF. To prove soundness it suffices to prove that subtyping is appropriately approximated, as stated by the following lemma. □

LEMMA G.7. *If $\Gamma \vdash_S \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$ then $\Gamma \vdash \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$.*

PROOF. By rule $\leq\text{-BASE-}\lambda^S$, we need to show that $\forall \theta \in \llbracket \Gamma \rrbracket. \llbracket \theta \cdot \{v : B \mid e_1\} \rrbracket \subseteq \llbracket \theta \cdot \{v : B \mid e_2\} \rrbracket$. We fix a $\theta \in \llbracket \Gamma \rrbracket$, and get that forall bindings $(x_i : \{v : B^\perp \mid e_i\}) \in \Gamma$, $\theta \cdot e_i[v/x_i] \hookrightarrow^* \text{True}$.

Then need to show that for each e , if $e \in \llbracket \theta \cdot \{v : B \mid e_1\} \rrbracket$, then $e \in \llbracket \theta \cdot \{v : B \mid e_2\} \rrbracket$.

If e diverges then the statement trivially holds. Assume $e \hookrightarrow^* w$. We need to show that if $\theta \cdot e_1[v/w] \hookrightarrow^* \text{True}$ then $\theta \cdot e_2[v/w] \hookrightarrow^* \text{True}$.

Let θ^\perp the lifted substitution that satisfies the above. Then by Lemma G.4 for each model $\sigma^\beta \in \llbracket \theta^\perp \rrbracket$, $\sigma^\beta \models p_i$, and $\sigma^\beta \models q_1$ for $\Gamma \vdash e_i[v/x_i] \rightsquigarrow p_i$, $\Gamma \vdash e_i[v/w] \rightsquigarrow q_i$. Since $\Gamma \vdash_S \{v : B \mid e_1\} \leq \{v : B \mid e_2\}$ we get

$$\bigwedge_i p_i \Rightarrow q_1 \Rightarrow q_2$$

thus $\sigma^\beta \models q_2$. By Theorem F.3 we get $\theta \cdot e_2[v/w] \hookrightarrow^* \text{True}$. □

H REASONING ABOUT LAMBDA

Encoding of λ -abstractions and applications via uninterpreted functions, while sound, is *imprecise* as it makes it hard to prove theorems that require α - and β -equivalence or extensional equality. Using the universally quantified α - and β -equivalence axioms would let the type checker accept more programs, but would render validity, and hence, type checking undecidable. Next, we identify a middle ground by describing an not provably complete, but sound and decidable approach to increase the precision of type checking by strengthening the VCs with instances of the α - and β -equivalence axioms § H.1 and by introducing a combinator for safely asserting extensional equality § H.2. In the sequel, we omit `app` when it is clear from the context.

H.1 Equivalence

As soundness relies on satisfiability under a σ^β (see Definition F.1), we can safely *instantiate* the axioms of α - and β -equivalence on any set of terms of our choosing and still preserve soundness (Theorem 5.2). That is, instead of checking the validity of a VC $p \Rightarrow q$, we check the validity of a *strengthened VC*, $a \Rightarrow p \Rightarrow q$, where a is a (finite) conjunction of *equivalence instances* derived from p and q , as discussed below.

Representation Invariant The lambda binders, for each SMT sort, are drawn from a pool of names x_i where the index $i = 1, 2, \dots$. When representing λ terms we enforce a *normalization invariant* that for each lambda term `lam x_i e`, the index i is greater than any lambda argument appearing in e .

α -instances For each syntactic term `lam x_i e` and λ -binder x_j such that $i < j$ appearing in the VC, we generate an *α -equivalence instance predicate* (or *α -instance*):

$$\text{lam } x_i e = \text{lam } x_j e[x_i/x_j]$$

The conjunction of α -instances can be more precise than De Bruijn representation, as they let the SMT solver deduce more equalities via congruence. For example, this VC is needed to prove the applicative laws for `Reader`:

$$d = \text{lam } x_1 (x x_1) \quad \Rightarrow \quad \text{lam } x_2 ((\text{lam } x_1 (x x_1)) x_2) = \text{lam } x_1 (d x_1)$$

The α instance `lam x_1 (d x_1) = lam x_2 (d x_2)` derived from the VC's hypothesis, combined with congruence immediately yields the VC's consequence.

β -instances For each syntactic term `app (lam x e) e_x` , with e_x not containing any λ -abstractions, appearing in the VC, we generate a *β -equivalence instance predicate* (or *β -instance*):

$$\text{app (lam } x_i e) e_x = e[e_x/x_i], \text{ s.t. } e_x \text{ is } \lambda\text{-free}$$

The λ -free restriction is a simple way to enforce that the reduced term $e[e_x/x_i]$ enjoys the representation invariant. For example, consider the following VC needed to prove that the bind operator for lists satisfies the monadic associativity law.

$$(f x \gg= g) = \text{app (lam } y (f y \gg= g)) x$$

The right-hand side of the above VC generates a β -instance that corresponds directly to the equality, allowing the SMT solver to prove the (strengthened) VC.

Normalization The combination of α - and β -instances is often required to discharge proof obligations. For example, when proving that the bind operator for the `Reader` monad is associative, we need to prove the VC:

$$\text{lam } x_2 (\text{lam } x_1 w) = \text{lam } x_3 (\text{app (lam } x_2 (\text{lam } x_1 w)) w)$$

The SMT solver proves the VC via the equalities corresponding to an α and then β -instance:

$$\text{lam } x_2 (\text{lam } x_1 w) =_{\alpha} \text{lam } x_3 (\text{lam } x_1 w) =_{\beta} \text{lam } x_3 (\text{app } (\text{lam } x_2 (\text{lam } x_1 w)) w)$$

H.2 Extensionality

Often, we need to prove that two functions are equal, given the definitions of reflected binders. Consider

```
reflect id
id x = x
```

LIQUID HASKELL accepts the proof that $\text{id } x = x$ for all x :

```
id_x_eq_x :: x:a → {id x = x}
id_x_eq_x = \x → id x =. x ** QED
```

as “calling” id unfolds its definition, completing the proof. However, consider this η -expanded variant of the above proposition:

```
type Id_eq_id = {(\x → id x) = (\y → y)}
```

LIQUID HASKELL *rejects* the proof:

```
fails :: Id_eq_id
fails = (\x → id x) =. (\y → y) ** QED
```

The invocation of id unfolds the definition, but the resulting equality refinement $\{\text{id } x = x\}$ is *trapped* under the λ -abstraction. That is, the equality is absent from the typing environment at the *top* level, where the left-hand side term is compared to $\lambda y \rightarrow y$. Note that the above equality requires the definition of id and hence is outside the scope of purely the α - and β -instances.

An Extensionality Operator To allow function equality via extensionality, we provide the user with a (family of) *function comparison operator(s)* that transform an *explanation* ρ which is a proof that $f x = g x$ for every argument x , into a proof that $f = g$.

```
=V :: f:(a → b) → g:(a → b) → exp:(x:a → {f x = g x}) → {f = g}
```

Of course, $=V$ cannot be implemented; its type is *assumed*. We can use $=V$ to prove Id_eq_id by providing a suitable explanation:

```
pf_id_id :: Id_eq_id
pf_id_id = (\y → y) =V (\x → id x) .: expl ** QED where expl = (\x → id x =. x
** QED)
```

The explanation is the second argument to \cdot : which has the following type that syntactically fires β -instances:

```
x:a → {(\x → id x) x = ((\x → x) x)}
```

I IMPLEMENTATION

Refinement reflection and PLE are implemented in Liquid Haskell. The implementation can be found in the [Liquid Haskell GitHub repository](#), all the benchmarks of § 2 and § 7 are included in the [nople](#) and [ple](#) test directories. The benchmarks for deterministic parallelism can be found at [class-laws](#) and [detpar-laws](#).

Next, we describe the file [ProofCombinators.hs](#), the library of proof combinators used by our benchmarks and discuss known limitations of our implementation.

1.1 ProofCombinators: The Proof Combinators Library

In this section we present ProofCombinators, a Haskell library used to structure proof terms. ProofCombinators is inspired by Equational Reasoning Data Types in Adga [Mu et al. 2009], providing operators to construct proofs for equality and linear arithmetic in Haskell. The constructed proofs are checked by an SMT-solver via Liquid Types.

Proof terms are defined in ProofCombinators as a type alias for unit, a data type that carries no run-time information

```
type Proof = ()
```

Proof types are refined to express theorems about program functions. For example, the following Proof type expresses that `fib 2 == 1`

```
fib2 :: () → {v:Proof | fib 2 == 1}
```

We simplify the above type by omitting the irrelevant basic type Proof and variable v

```
fib2 :: () → { fib 2 == 1 }
```

ProofCombinators provides primitives to construct proof terms by casting expressions to proofs. To resemble mathematical proofs, we make this casting post-fix. We write `p *** QED` to cast p to a proof term, by defining two operators QED and *** as

```
data QED = QED
```

```
(***) :: a → QED → Proof
_ *** _ = ()
```

Proof construction. To construct proof terms, ProofCombinators provides a proof constructor \odot . for logical operators of the theory of linear arithmetic and equality: $\{=, \neq, \leq, <, \geq, >\} \in \odot$. $\odot. x y$ ensures that $x \odot y$ holds, and returns x

```
 $\odot$ :: x:a → y:{a | x  $\odot$  y} → {v:a | v==x}
 $\odot. x _ = x$ 
```

```
-- for example
```

```
==. :: x:a → y:{a | x==y} → {v:a | v==x}
```

For instance, using `==.` we construct a proof, in terms of Haskell code, that `fib 2 == 1`:

```
fib2 _
= fib 2
==. fib 1 + fib 0
==. 1
*** QED
```

Reusing proofs: Proofs as optional arguments. Often, proofs require reusing existing proof terms. For example, to prove `fib 3 == 2` we can reuse the above `fib2` proof. We extend the proof combinators, to receive an *optional* third argument of Proof type.

```
 $\odot$ :: x:a → y:a → {x  $\odot$  y} → {v:a | v==x}
 $\odot. x _ _ = x$ 
```

$\odot. x y p$ returns x while the third argument p explicitly proves $x \odot y$.

Optional Arguments. The proof term argument is optional. To implement optional arguments in Haskell we use the standard technique where for each operator \odot we define a type class `Opt \odot` that

takes as input two expressions a and returns a result r , which will be instantiated with either the result value $r := a$ or a function form a proof to the result $r := \text{Proof} \rightarrow a$.

```
class Opt⊙ a r where
  (⊙.) :: a → a → r
```

When no explicit proof argument is required, the result type is just an $y : a$ that carries the proof $x \odot y$

```
instance Opt⊙ a a where
  (⊙.) :: x : a → y : {a | x ⊙ y} → {v : a | v == x }
  (⊙.) x _ = x
```

Note that Haskell's type inference [Sulzmann et al. 2006] requires both type class parameters a and r to be constrained at class instance matching time. In most our examples, the result type parameter r is not constrained at instance matching time, thus due to the Open World Assumption the matching instance could not be determined. To address the above, we used another common Haskell trick, of generalizing the instance to type arguments a and b and then constraint a and b to be equal $a \sim b$. This generalization allows the instance to always match and imposed the equality constraint after matching.

```
instance (a ~ b) ⇒ Opt⊙ a b where
  (⊙.) :: x : a → y : {x ⊙ y} → {v : b | v == x }
  (⊙.) x _ = x
```

To explicitly provide a proof argument, the result type r is instantiated to $r := \text{Proof} \rightarrow a$. For the same instance matching restrictions as above, the type is further generalized to return some b that is constraint to be equal to a .

```
instance (a ~ b) ⇒ Opt⊙ a (Proof → b) where
  (⊙.) :: x : a → y : a → {x ⊙ y} → {v : b | v == x }
  (⊙.) x _ _ = x
```

As a concrete example, we define the equality operator `==`. via the type class `OptEq` as

```
class OptEq a r where
  (==.) :: a → a → r

instance (a ~ b) ⇒ OptEq a b where
  (==.) :: x : a → y : {a | x == y} → {v : b | v == x }
  (==.) x _ = x

instance (a ~ b) ⇒ OptEq a (Proof → b) where
  (==.) :: x : a → y : a → {x == y} → {v : b | v == x }
  (==.) x _ _ = x
```

Explanation Operator. The “explanation operator” `(?)`, or `(·?)`, is used to better structure the proofs. `(?)` is an infix operator with same fixity as `(⊙.)` that allows for the equivalence $x \odot. y ? p == (\odot.) x y p$

```
(?) :: (Proof → a) → Proof → a
f ? y = f y
```

Putting it all together Using the above operators, we prove that `fib 3 == 2`, reusing the previous proof of `fib 2 == 1`, in a Haskell term that resembles mathematical proofs

```

fib3 :: () → {fib 3 == 2}
fib3 _
  =   fib 3
  ==. fib 2 + fib 1
  ==. 2           ? fib2 ()
  *** QED

```

Unverified Operators All operators in ProofCombinators, but two are implemented in Haskell with implementations verified by Liquid Haskell. The “unsound” operators are the assume (1). (==?) that eases proof construction by assuming equalities, to be proven later and (2). ($\text{=}\forall$) extensional proof equality.

Assume Operator (==?) eases proof construction by assuming equalities while the proof is in process. It is not implemented in that its body is `undefined`. Thus, if we run proof terms including assume operator, the proof will merely crash (instead of returning `()`). Proofs including the assume operator are not considered complete, as via assume operator any statement can be proven,

Function Extensional Equality Unlike the assume operator that is undefined and included in unfinished thus unsound proofs, the functions extensionality is included in valid proofs that assume function extensionality, an axioms that is assumed, as it cannot be proven by our logic.

To allow function equality via extensionality, we provide the user with a function comparison operator that for each function f and g it transforms a proof that for every argument x , $f\ x = g\ x$ to a proof on function equality $f = g$.

```

(=V) :: Arg a ⇒ f:(a → b) → g:(a → b)
      → p:(x:a → {f x = g x})
      → {f = g}

```

The function ($\text{=}\forall$) is not implemented in the library: it returns `()` and its type is assumed. But soundness of its usage requires the argument type variable a to be constrained by a type class constraint `Arg a`, for both operational and type theoretic reasons.

From *operational* point of view, an implementation of ($\text{=}\forall$) would require checking equality of $f\ x = g\ x$ *forall* arguments x of type a . This equality would hold due to the proof argument p . The only missing point is a way to enumerate all the argument a , but this could be provided by a method of the type class `Arg a`. Yet, we have not implement ($\text{=}\forall$) because we do not know how to provide such an implementation that can provably satisfy ($\text{=}\forall$)’s type.

From *type theoretic* point of view, the type variable argument a appears only on negative positions. Liquid type inference is smart enough to infer that since a appears only negative ($\text{=}\forall$) cannot use any a and thus will not call any of its argument arguments f , g , nor the p . Thus, at each call site of ($\text{=}\forall$) the type variable ‘ a ’ is instantiated with the refinement type $\{v:a \mid \text{false}\}$ indicating dead-code (since a will not be used by the callee.) Refining the argument $x:a$ with `false` at each call-site though leads to unsoundness, as each proof argument p is a valid proof under the `false` assumption. What Liquid inference cannot predict is our intention to call f , g and p at *every possible argument*. This information is capture by the type class constraint `Arg a` that (as discussed before [Vazou et al. 2013]) states that methods of the type class `Arg a` may create values of type a , thus, due to lack of information on the values that are created by the methods of `Arg a`, a can only be refined with `True`.

With extensional equality, we can prove that $\lambda x \rightarrow x$ is equal to $\lambda x \rightarrow \text{id}\ x$, by providing an explicit explanation that if we call both these functions with the same argument x , they return the same result, for each x .

```

safe :: Arg a ⇒ a

```

```

    → {(\x → id x) = (\x → x)}
safe _ = (\x → x)
      =V(\x → id x) ∴ (exp ())

exp :: Arg a ⇒ a → x:a
    → {(\x → id x) x = (\x → x) x}
exp _ x = id x
      ==. x
      *** QED

```

Note that the result of `exp` is an equality of the redexes $(\lambda x \rightarrow \text{id } x) x$ and $((\lambda x \rightarrow x) x)$. Extensional function equality requires as argument an equality on such redexes. Via β equality instantiations, both such redexes will automatically reduce, requiring `exp` to prove $\text{id } x = x$, with `is direct`.

Admittedly, proving function equality via extensionality is requires a cumbersome indirect proof. For each function equality in the main proof one needs to define an explanation function that proves the equality for every argument.

1.2 Engineering Limitations

The theory of refinement reflection is fully implemented in Liquid Haskell. Yet, to make this extension *usable* in *real world applications* there are four known engineering limitations that need to be addressed. All these limitations seem straightforward to address and we plan to fix them soon.

The language of refinements is typed lambda calculus. That is the types of the lambda arguments are explicitly specified instead of being inferred. As another minor limitation, the refinement language parser requires the argument to be enclosed in parenthesis in applications where the function is not a variable. Thus the Haskell expression $(\lambda x \rightarrow x) e$ should be written as $(\lambda x:a \rightarrow x) (e)$ in the refinement logic,

Class instances methods can not be reflected. Instead, the methods we want to use in the theorems/propositions should be defined as Haskell functions. This restriction has two major implications. Firstly, we can not verify correctness of library provided instances but we need to redefine them ourselves. Secondly, we cannot really verify class instances with class preconditions. For example, during verification of monoid associativity of the `Maybe` instance

```
instance (Monoid a) ⇒ Monoid (Maybe a)
```

there is this `Monoid a` class constraint assumption we needed to raise to proceed verification.

Only user defined data types can currently used in verification. The reason for this limitation is that reflection of case expressions requires checker and projector measures for each data type used in reflected functions. Thus, not only should these data types be defined in the verified module, but also should be be injected in the logic by providing a refined version of the definition that can (or may not) be trivially refined.

For example, to reflect a function that uses `Peano` numbers, the Haskell *and* the refined `Peano` definitions should be provided

```
data Peano = Z | S Peano

{-@ data Peano [toInt]
    = Z
    | S {prev :: Peano}

```

```
@-}
```

Note that the termination function `toInt` that maps `Peano` numbers to natural numbers is also crucial for soundness of reflection.

There is no module support. All reflected definitions, including, measures (automatically generated checkers and selector, but also the classic lifted Haskell functions to measures) and the reflected types of the reflected functions, are not exposed outside of the module they are defined. Thus all definitions and propositions should exist in the same module.

J VERIFIED DETERMINISTIC PARALLELISM

Finally, we evaluate our deterministic parallelism prototypes. Aside from the lines of proof code added, we evaluate the impact on runtime performance. Were we using a proof tool external to Haskell, this would not be necessary. But our proofs are Haskell programs—they are necessarily visible to the compiler. In particular, this means a proliferation of unit values and functions returning unit values. Also, typeclass instances are witnessed at runtime by “dictionary” data structures passed between functions. Layering proof methods on top of existing classes like `Ord` could potentially add indirection or change the code generated, depending on the details of the optimizer. In our experiments we find little or no effect on runtime performance. Benchmarks were run on a single-socket Intel® Xeon® CPU E5-2699 v3 with 18 physical cores and 64GiB RAM.

J.1 LVish: Concurrent Sets

First, we use the `verifiedInsert` operation to observe the runtime slowdown imposed by the extra proof methods of `VerifiedOrd`. We benchmark concurrent sets storing 64-bit integers. Figure 18 compares the parallel speedups for a fixed number of parallel insert operations against parallel `verifiedInsert` operations, varying the number of concurrent threads. There is a slight observable difference between the two lines because the extra proof methods do exist at runtime. We repeat the experiment for two set implementations: a concurrent skiplist (`SLSet`) and a purely functional set inside an atomic reference (`PureSet`) as described in Kuper et al. [2014].

J.2 monad-par: n-body simulation

Next, we verify deterministic behavior of an n -body simulation program that leverages `monad-par`, a Haskell library which provides deterministic parallelism for pure code [Marlow et al. 2011].

Each simulated particle is represented by a type `Body` that stores its position, velocity, and mass. The function `accel` computes the relative acceleration between two bodies:

```
accel :: Body → Body → Accel
```

where `Accel` represents the three-dimensional acceleration

```
data Accel = Accel Real Real Real
```

To compute the total acceleration of a body `b` we (1) compute the relative acceleration between `b` and each body of the system (`Vec Body`) and (2) we add each acceleration component. For efficiency, we use a parallel `mapReduce` for the above computation that first *maps* each vector body to get the acceleration relative to `b` (`accel b`) and then adds each `Accel` value by pointwise addition. `mapReduce` is only deterministic if the element is a `VerifiedMonoid`.

```
mapReduce :: VerifiedMonoid b ⇒ (a → b) → Vec a → b
```

To enforce the determinism of an n -body simulation, we need to provide a `VerifiedMonoid` instance for `Accel`. We can prove that $(\text{Real}, +, \mathbf{0}, \mathbf{0})$ is a monoid. By product proof composition, we get a verified monoid instance for

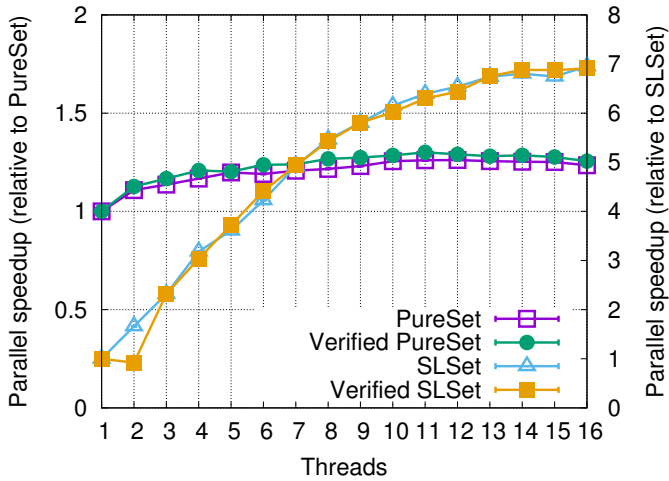


Fig. 18. Parallel speedup for doing 1 million parallel inserts over 10 iterations, verified and unverified, relative to the unverified version, for PureSet and SLSet.

```
type Accel' = (Real, (Real, Real))
```

which is isomorphic to `Accel` (i.e. `Iso Accel' Accel`).

Figure 19 shows the results of running two versions of the n -body simulation with 2,048 bodies over 5 iterations, with and without verification, using floating point doubles for `Real`¹. Notably, the two programs have almost identical runtime performance. This demonstrates that even when verifying code that is run in a tight loop (like `accel`), we can expect that our programs will not be slowed down by an unacceptable amount.

J.3 DPJ: Parallel Reducers

The Deterministic Parallel Java (DPJ) project provides a deterministic-by-default semantics for the Java programming language [Bocchino et al. 2009]. In DPJ, one can declare a method as commutative and thus *assert* that racing instances of that method result in a deterministic outcome. For example:

```
commutative void updateSum(int n) writes R { sum += n; }
```

But, DPJ provides no means to formally prove commutativity and thus determinism of parallel reduction. In Liquid Haskell, we specified commutativity as an extra proof method that extends the `VerifiedMonoid` class.

```
class VerifiedMonoid a => VerifiedCommutativeMonoid a where
  commutes :: x:a -> y:a -> { x <> y = y <> x }
```

Provably commutative appends can be used to deterministically update a reducer variable, since the result is the same regardless of the order of appends. We used `LVish` [Kuper et al. 2014] to encode a reducer variable with a value `a` and a region `s` as `RVar s a`.

```
newtype RVar s a
```

We specify that safe (i.e. deterministic) parallel updates require provably commutative appending.

¹Floating point numbers notoriously violate associativity, but we use this approximation because Haskell does not yet have an implementation of *superaccumulators* [Collange et al. 2014].

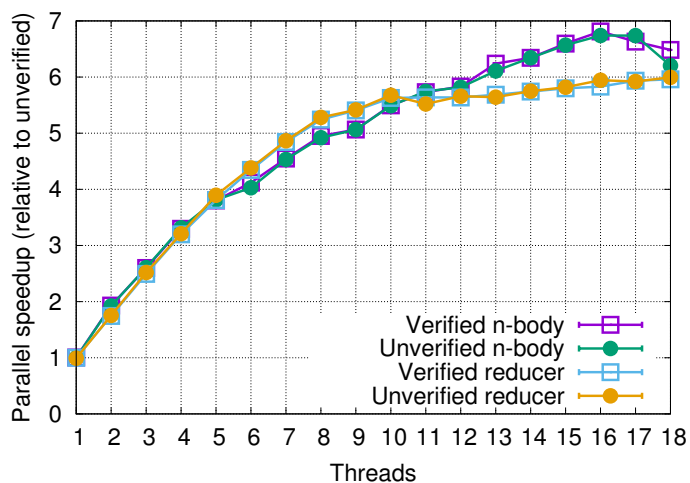


Fig. 19. Parallel speedup for doing a parallel n -body simulation and parallel array reduction. The speedup is relative to the unverified version of each respective class of program.

```
updateRVar :: VerifiedCommutativeMonoid a => a -> RVar s a -> Par s ()
```

Following the DPJ program, we used `updateRVar`'s provably deterministic interface to compute, in parallel, the sum of an array with 3×10^9 elements by updating a single, global reduction variable using a varying number of threads. Each thread sums segments of an array, sequentially, and updates the variable with these partial sums. In Figure 19, we compare the verified and unverified versions of our implementation to observe no appreciable difference in performance.