Gradual Liquid Type Inference

Abstract

We present gradual liquid type inference, a novel combination of refinement types with gradual refinements that range over a finite set of SMT-decidable predicates. This finiteness restriction allows for an algorithmic inference procedure where all possibly valid interpretations of a gradual refinement are exhaustively checked. Thanks to exhaustive searching we can detect the *safe concretizations*, *i.e.* the concrete refinements that justify why a program with gradual refinements is well-typed. We make the novel observation that gradual liquid type inference can be used for static liquid type error explanation, since the safe concretizations exhibit all the potential inconsistencies that lead to type errors. Based on Liquid Haskell, we implement gradual liquid type inference in `GuiLT`, a tool that interactively presents all the safe concretizations of gradual liquid types and demonstrate its utility for user-guided migration of three commonly-used Haskell list manipulation libraries.

1 Introduction

Refinement types [3] allow for lightweight program verification by decorating existing program types with logical predicates. For instance, the type `{x:Int | 0 < x}` denotes strictly positive integer values and can be used to validate *at compile time* the absence of division-by-zero errors by refining the type of division:

```
(/) :: Int → {x:Int | 0 < x} → Int
```

A major challenge with refinement types is to support decidable automatic checking and *inference*. To this end, *liquid types* restrict refinement predicates to decidable theories [16]. The type `{x:Int | k}` describes integer values refined with some predicate `k`, that is automatically solved based on unifying the constraints generated at each use of `x`, resulting in a concrete refinement drawn from a *finite* domain of template refinements. The attractiveness of liquid types for programmers is usability. Verification only requires specification of top-level functions, while all intermediate types can be automatically inferred and checked.

However, liquid types, as most type inference approaches, suffer from terrible error messages [27, 29]. At an ill-typed function application, the system should decide whether to blame the function definition or the client; the particular choice will unfortunately and inevitably expose some of the internals of the sophisticated verification procedure on to the user. The difficulty of understanding error messages from liquid inference in turn makes it really hard to progressively migrate non-refined code, say in Haskell, to refined code, say in Liquid Haskell [25].

For instance, consider the following function `divIf` that either inverts its argument `x` if it is positive, or `1-x` otherwise:

```
divIf x = if isPos x then 1/x else 1/(1-x)
```

The function relies on an imported function `isPos` of type `Int → Bool`. If we want to give `divIf` a refined type, liquid inference starts from the signature:

```
divIf :: x:{ Int | k_x } → {o:Int | k_o }
```

and solves the variables $\mathtt{k}_x$ and $\mathtt{k}_o$ based on the use sites. However, without any uses of `divIf` in the considered source code, and for reasons we will clarify in due course, liquid inference infers useless refinements:

```
divIf :: x:{ Int | false } → {o:Int | false }
```

The false refinement on the argument means any future *client* will be rejected. Conversely, if the code base at the time inference is run includes a "positive" client call `divIf 1`, the inferred precondition would be `0 < x`, triggering a type error in the *definition* of `divIf` due to the lack of information about `isPos`. This hard-to-predict and moving blame is not unique to liquid type inference; it frequently appears in type inference engines, yielding hard-to-debug error messages.

A key contribution of this paper is to recognize that, in such situations, treating the inferred refinement as an *unknown* refinement—in the sense of gradual typing [5, 18]—allows us to explain liquid type errors. Specifically, we adapt the gradual refinement types of Lehmann and Tanter [8] to the setting of liquid type inference, yielding gradual liquid type inference. Programmers can introduce gradual refinements such as `{x: Int | ?}` and inference exhaustively searches for *safe concretizations* (SCs for short), *i.e.* the concrete refinements that can replace each occurrence of a gradually-refined variable to make the program well-typed. These SCs can then be used to understand liquid type errors and assist in migrating programs to adopt liquid types.

Contributions

- We give a semantics and inference algorithm for gradual liquid types (§ 4). We prove that inference is correct and satisfies the static criteria for gradual languages [20].
- We implement gradual liquid type inference in `GuiLT`, as an extension of Liquid Haskell (§ 5).
- `GuiLT` takes as input a Haskell program annotated with gradual refinements and generates an interactive program that presents all valid choices to replace each `?`. The user can explore suggested predicates and decide which ones to use so that their program is well-typed (§ 6).
- We use `GuiLT` for user-guided migration of three existing Haskell libraries (1260 LoC) to Liquid Haskell (§ 7), demonstrating gradual liquid type inference can be effictively supported and used interactively for program migration.

2 Overview

We start by summarizing gradual liquid type inference: the intersection of gradual refinements (§ 2.4) with liquid types (§ 2.2), *i.e.* a decidable fragment of refinement types (§ 2.1).

2.1 Refinement Types

To explain refinement type checking, let us consider the following type signatures for the example of § 1:

```
isPos :: x:Int → {b:Bool | b ⇔ 0 < x}
divIf :: Int → Int
```

With these specifications, `divIf` is well-typed because the precondition of `(/)` is provably satisfied. Refinement type checking proceeds in three steps, described below.

Step 1: Constraint Generation Based on the code and the specifications, refinement subtyping constraints are generated; for our example, two subtyping constraints are generated, one for each call to `(/)`. They stipulate that, in the environment with the argument `x` and the boolean branching guard `b`, the second argument to `(/)` (*i.e.* `v = x` and `v = 1-x`, *resp.*) is *safe*, *i.e.* it respects the precondition `0 < v`.

```
b:{b ⇔ 0 < x ∧ b} ⊢ {v | v = x}   ≤ {v | 0 < v}
b:{b ⇔ 0 < x ∧ ¬b} ⊢ {v | v = 1-x} ≤ {v | 0 < v}
```

For space, we write `{v | p}` to denote `{v:t | p}` when the type `t` is clear; we omit the refinement variables from the environment, simplifying `x:{x | p}` to `x:{p}`, and we skip uninformative refinements such as `x:{true}`.

In both constraints the branching guard `b` is refined with the result refinement of `isPos`: `b ⇔ 0 < x`. Also, the environment is strengthened with the value of the condition in each branch: `b` in the **`then`** branch, `¬ b` in the **`else`** branch.

Step 2: Verification Conditions Each subtyping constraint is reduced to a logical verification condition (VC), that validates if, assuming all the refinements in the environment, the refinement on the left-hand side implies the one on the right-hand side. For instance, the two constraints above reduce to the following VCs.

```
b ⇔ 0 < x ∧  b ⇒ v = x   ⇒ 0 < v
b ⇔ 0 < x ∧ ¬b ⇒ v = 1-x ⇒ 0 < v
```

Step 3: Implication Checking Finally, an SMT solver is used to check the validity of the generated VCs, and thus determine if the program is well-typed. Here, the SMT solver determines that both VCs are valid, thus `divIf` is well-typed.

2.2 Liquid Types

When not all refinement types are spelled out explicitly, one can use *inference*. For instance, the well-typedness of `divIf` crucially relies on the guard predicate, as propagated by the refinement type of `isPos`. We now explain how liquid typing [16] infers a type for `divIf` in case `isPos` is an imported, *unrefined* function. Liquid types introduce refinement type variables, known as *liquid variables*, for unspecified refinements; here k_x and k_o (§ 1). After generating subtyping constraints as in § 2.1, the inference procedure attempts to find a solution for the liquid variables such that all the constraints are satisfied. If no solution can be found, the program is deemed ill-typed.

Step 1: Constraint Generation After introduction of the liquid variables, the following subtyping constraints are generated for `divIf`.

$$\texttt{x:\{}\mathsf{k}_x\texttt{\}, b:\{b\}} \vdash \texttt{\{v | v=x \}} \leq \texttt{\{v | 0 < v\}}$$
$$\texttt{x:\{}\mathsf{k}_x\texttt{\}, b:\{}\neg\texttt{b\}} \vdash \texttt{\{v | v=1-x\}} \leq \texttt{\{v | 0 < v\}}$$

Step 2: Constraint Solving Liquid inference then solves the liquid variables `k` so that the subtyping constraints are satisfied. The solving procedure takes as input a *finite* set of refinement *templates* $\mathbb{Q}^\star$ abstracted over program variables. For example, the template set $\mathbb{Q}^\star$ below describes ordering predicates, with $\star$ ranging over program variables.

$$\mathbb{Q}^\star = \{0 < \star,\ 0 \leq \star,\ \star < 0,\ \star \leq 0,\ \star < \star,\ \star \leq \star\}$$

Next, for each liquid variable, the set $\mathbb{Q}^\star$ is instantiated with all program variables in scope, to generate well-sorted predicates. Instantiation of $\mathbb{Q}^\star$ for the liquid variables k_x and k_o leads to the following concrete predicate candidates.

$$\mathbb{Q}^x = \{\ 0{<}x,\ 0{\leq}x,\ x{<}0,\ x{\leq}0\ \}$$
$$\mathbb{Q}^o = \{\ 0{<}o,\ 0{\leq}o,\ o{<}0,\ o{\leq}0,\ o{<}x,\ x{<}o,\ \ldots\ \}$$

Finally, inference iteratively computes the strongest solution for each liquid variable that satisfies the constraints. It starts from an initial solution that maps each variable to the logical conjunction of all the instantiated templates

$$A = \{\mathsf{k}_x \mapsto \bigwedge \mathbb{Q}^x,\ \mathsf{k}_o \mapsto \bigwedge \mathbb{Q}^o\}$$

It then repeatedly filters out predicates of the solution until all constraints are satisfied.

In our example, the initial solution—which includes the contradictory predicates $0 < x$ and $x < 0$— solves both liquid variables to false. As discussed in 1, this inferred solution is however useless in practice. With client code that imposes additional constraints, the inferred solution can be more useful, though the reported errors will be hard to interpret.

2.3 Gradual Refinement Types

We observe that we can exploit gradual typing in order to assist inference and provide better support for error explanation and program migration. Instead of interpreting an unspecified refinement as a liquid variable, let us use the unknown gradual refinement `{Int | ?}` for the argument type of `divIf` [8]. This precondition specifies that for each *usage occurrence* of the argument `x`, there must *exist* a concrete refinement (which we call a *safe concretization*, SC) for which the (non-gradual) program type checks. Key to this definition is that the refinement that exists need not be unique to all occurrences of the identifier. Gradual refinement type checking proceeds as follows.

Step 1: Constraint Generation First, we generate the subtyping constraints derived from the definition of `divIf` that now contain gradual refinements.

```
x:{?}, b:{b}  ⊢ {v | v=x  } ≤ {v | 0 < v}
x:{?}, b:{¬b} ⊢ {v | v=1-x} ≤ {v | 0 < v}
```

Step 2: Gradual Verification Conditions Each subtyping reduces to a VC, where each gradual refinement such as `x:{?}` translates intuitively to an existential refinement (`∃ p. p x`). The solution of these existentials are the safe concretizations (SCs) of the program. Here, we informally use $\exists^?$ `p` to denote such existentials over predicates, and call the corresponding verification conditions *gradual VCs* (GVCs). For example, the GVCs for `divIf` are the following.

$$(\exists^?\ \mathtt{p_{then}}.\ \mathtt{p_{then}\ x})\ \wedge\ \mathtt{b}\ \Rightarrow\ \mathtt{v{=}x}\ \Rightarrow\ \mathtt{0 < v}$$
$$(\exists^?\ \mathtt{p_{else}}.\ \mathtt{p_{else}\ x})\ \wedge\ \neg\mathtt{b}\ \Rightarrow\ \mathtt{v{=}1{-}x}\ \Rightarrow\ \mathtt{0 < v}$$

Step 3: Gradual Implication Checking Checking the validity of the generated GVCs is an open problem. In the `divIf` example, we can, by observation, find the SCs that render the GVCs valid: $\mathtt{p_{then}\ x} \mapsto \mathtt{0 < x}$ and $\mathtt{p_{else}\ x} \mapsto \mathtt{x} \le \mathtt{0}$.

More importantly, we can present these solutions to the user as the conditions under which `divIf` is well-typed.

Our goal is to find an algorithmic procedure that solves GVCs. Lehmann and Tanter [8] describe how GVCs over linear arithmetic can be checked, while Courcelle and Engelfriet [2] describe a more general logical fragment (monadic second order logic) with an algorithmic decision procedure. Yet, in both cases we lose the justifications, *i.e.* the SCs, and thus the opportunity to use such SCs for error explanation and migration assistance.

2.4 Gradual Liquid Type Inference

To algorithmically solve GVCs we can exhaustively search for SCs in the *finite* predicate domain of liquid types. In between constraint generation and constraint solving, gradual liquid type inference concretizes the constraints by instantiating gradual refinements with each possible liquid template.

Step 1: Constraint Generation Constraint generation is performed exactly like gradual refinement types, leading to the constraints of § 2.3 for the `divIf` example.

Step 2: Constraint Concretization We exhaustively generate all the possible concretizations of the constraints. For example, `x:{?}` can be concretized to any predicate from the $\mathbb{Q}^x$ set of § 2.2, yielding $|\mathbb{Q}^x|^2 = 16$ concrete constraint sets, among which the following two.

1. **Concretization for** $\mathtt{p_{then}\ x} \mapsto \mathtt{0{<}x}$, $\mathtt{p_{else}\ x} \mapsto \mathtt{0{<}x}$:
   ```
   x:{0<x}, b:{b}  ⊢ {v | v=x  } ≤ {v | 0<v}
   x:{0<x}, b:{¬b} ⊢ {v | v=1-x} ≤ {v | 0<v}
   ```
2. **Concretization for** $\mathtt{p_{then}\ x} \mapsto \mathtt{0{<}x}$, $\mathtt{p_{else}\ x} \mapsto x \le 0$:
   ```
   x:{0<x},  b:{b}  ⊢ {v | v=x  } ≤ {v | 0<v}
   x:{x≤0}, b:{¬b} ⊢ {v | v=1-x} ≤ {v | 0<v}
   ```

Constants	c	$::=$	$\wedge \mid \neg \mid = \mid \ldots$
		$\mid$	$\mathtt{true} \mid \mathtt{false}$
		$\mid$	$0, 1, -1, \ldots$
Values	v	$::=$	$c \mid \lambda x.e$
Expressions	e	$::=$	$v \mid x \mid e\ x$
		$\mid$	$\mathtt{if}\ x\ \mathtt{then}\ e\ \mathtt{else}\ e$
		$\mid$	$\mathtt{let}\ x = e\ \mathtt{in}\ e$
		$\mid$	$\mathtt{let}\ x{:}t = e\ \mathtt{in}\ e$
Predicates	p	$::=$	e
Basic Types	b	$::=$	$\mathtt{int} \mid \mathtt{bool}$
Types	t	$::=$	$\{x{:}b \mid p\} \mid x{:}t \to t$
Environment	Γ	$::=$	$\cdot \mid \Gamma, x{:}t$
Substitution	σ	$::=$	$\cdot \mid \sigma, (x, e)$
Constraint	C	$::=$	$\Gamma \vdash \{v{:}b \mid p\}$
		$\mid$	$\Gamma \vdash \{v{:}b \mid p\} \preceq \{v{:}b \mid p\}$

Figure 1. Syntax of λ^e_R.

Step 3: Constraint Solving After concretization, liquid constraint solving finds out the valid ones. In our example, the constraint 1 above is invalid while 2 is valid. Out of the 16 concrete constraints, only two are valid, with $\mathtt{p_{then}\ x} \mapsto \mathtt{0{<}x}$ and $\mathtt{p_{else}\ x} \mapsto x \le 0$ or $\mathtt{p_{else}\ x} \mapsto x < 0$. Thus, `divIf` type checks and also the inference provides to the user the SCs of $\mathtt{p_{then}}$ and $\mathtt{p_{else}}$ as an explanation of type checking.

In § 4 we formalize these three inference steps and prove the correctness and the gradual criteria of our algorithm. Various implementation considerations necessary for the algorithm to scale are described in § 5. We report on its use for error explanation and program migration in § 6 and § 7.

3 Liquid Types and Gradual Refinements

We briefly provide the technical background required to describe gradual liquid types. We start with the semantics and rules of a generic refinement type system (§ 3.1) which we then adjust to describe both liquid types (§ 3.2) and gradual refinement types (§ 3.3).

3.1 Refinement Types

Syntax Figure 1 presents the syntax of a standard functional language with refinement types, λ^e_R. The expressions of the language include constants, lambda terms, variables, function applications, conditionals, and let bindings. Note that the argument of a function application needs to be syntactically a variable, as must the condition of a conditional; this normalization is standard in refinement types as it simplifies the formalization [16]. There are two let binding forms, one where the type of the bound variable is inferred and one where it is explicitly declared.

λ^e_R types include *base refinements* $\{x{:}b \mid p\}$ where b is a base type (`int` or `bool`) refined with the logical predicate p. A predicate can be any expression e, which can refer to x. Types also include dependent function types $x{:}t_x \to t$, where x is bound to the function argument and can appear

Typing $\boxed{\Gamma \vdash e{:}t}$

$$\frac{\Gamma(x) = \{v{:}b \mid \_\}}{\Gamma \vdash x{:}\{v{:}b \mid v = x\}}\ \textsc{T-Var-Base} \qquad \frac{\Gamma(x) \text{ is a function type}}{\Gamma \vdash x{:}\Gamma(x)}\ \textsc{T-Var} \qquad \frac{}{\Gamma \vdash c{:}ty(c)}\ \textsc{T-Const}$$

$$\frac{\Gamma \vdash e{:}t_e \quad \Gamma \vdash t \quad \Gamma \vdash t_e \preceq t}{\Gamma \vdash e{:}t}\ \textsc{T-Sub} \qquad \frac{\Gamma, x{:}t_x \vdash e{:}t \quad \Gamma \vdash x{:}t_x \to t}{\Gamma \vdash \lambda x.e{:}(x{:}t_x \to t)}\ \textsc{T-Fun}$$

$$\frac{\Gamma \vdash e{:}(x{:}t_x \to t) \quad \Gamma \vdash y{:}t_x}{\Gamma \vdash e\ y{:}t[y/x]}\ \textsc{T-App} \qquad \frac{\begin{array}{c}\text{fresh } x' \quad \Gamma_1 \doteq \Gamma, x'{:}\{v{:}\texttt{bool} \mid x\} \quad \Gamma_2 \doteq \Gamma, x'{:}\{v{:}\texttt{bool} \mid \neg x\} \\ \Gamma \vdash x{:}\{v{:}\texttt{bool} \mid \_\} \quad \Gamma_1 \vdash e_1{:}t \quad \Gamma_2 \vdash e_2{:}t \quad \Gamma \vdash t\end{array}}{\Gamma \vdash \texttt{if } x \texttt{ then } e_1 \texttt{ else } e_2{:}t}\ \textsc{T-If}$$

$$\frac{\Gamma \vdash e_x{:}t_x \quad \Gamma, x{:}t_x \vdash e{:}t \quad \Gamma \vdash t}{\Gamma \vdash \texttt{let } x = e_x \texttt{ in } e{:}t}\ \textsc{T-Let} \qquad \frac{\Gamma \vdash e_x{:}t_x \quad \Gamma, x{:}t_x \vdash e{:}t \quad \Gamma \vdash t \quad \Gamma \vdash t_x}{\Gamma \vdash \texttt{let } x{:}t_x = e_x \texttt{ in } e{:}t}\ \textsc{T-Spec}$$

Sub-Typing $\boxed{\Gamma \vdash t \preceq t}$

$$\frac{\text{isValid}(\Gamma \vdash \{v{:}b \mid p_1\} \preceq \{v{:}b \mid p_2\})}{\Gamma \vdash \{v{:}b \mid p_1\} \preceq \{v{:}b \mid p_2\}}\ \textsc{S-Base} \qquad \frac{\Gamma \vdash t_{x2} \preceq t_{x1} \quad \Gamma, x{:}t_{x2} \vdash t_1 \preceq t_2}{\Gamma \vdash x{:}t_{x1} \to t_1 \preceq x{:}t_{x2} \to t_2}\ \textsc{S-Fun}$$

Well-Formedness $\boxed{\Gamma \vdash t}$

$$\frac{\text{isValid}(\Gamma \vdash \{v{:}b \mid p\})}{\Gamma \vdash \{v{:}b \mid p\}}\ \textsc{W-Base} \qquad \frac{\Gamma \vdash t_x \quad \Gamma, x{:}t_x \vdash t}{\Gamma \vdash x{:}t_x \to t}\ \textsc{W-Fun}$$

Figure 2. Static Semantics of λ^e_R. (Types colored in blue need to be inferred.)

in the result type t. As usual, we write b as a shortcut for $\{x{:}b \mid \texttt{true}\}$, and $t_x \to t$ as a shortcut for $x{:}t_x \to t$ when x does not appear in t.

Denotations Following Knowles and Flanagan [7], each type of λ^e_R denotes a set of expressions. The denotation of a base refinement includes all expressions that either diverge or evaluate to base values that satisfy the associated predicate. We write $\models e$ to represent that e is (operationally) valid and $e \Downarrow$ to represent that e terminates:

$$\models e \ \doteq\ e \hookrightarrow^\star \texttt{true} \qquad e \Downarrow \doteq \exists v.e \hookrightarrow^\star v$$

where $\cdot \hookrightarrow^\star \cdot$ is the reflexive, transitive closure of the small-step reduction relation. Denotations are naturally extended to function types and environments (as sets of substitutions).

$$\begin{aligned}
[\![\{x{:}b \mid p\}]\!] &\doteq \{e \mid \vdash e : b, \text{if } e \Downarrow \text{ then } \models p[e/x]\} \\
[\![x{:}t_x \to t]\!] &\doteq \{e \mid \forall e_x \in [\![t_x]\!].e\ e_x \in [\![t[e_x/x]]\!]\} \\
[\![\Gamma]\!] &\doteq \{\sigma \mid \forall x{:}t \in \Gamma.(x, e) \in \sigma \wedge e \in [\![\sigma \cdot t]\!]\}
\end{aligned}$$

Static semantics Figure 2 summarizes the standard typing rules that characterize whether an expression belongs to the denotation of a type [7, 16]. Namely, $e \in [\![t]\!]$ *iff* $\vdash e{:}t$. We define three kinds of relations 1. typing, 2. subtyping, and 3. well-formedness.

1. Typing: $\Gamma \vdash e{:}t$ *iff* $\forall \sigma \in [\![\Gamma]\!].\sigma \cdot e \in [\![\sigma \cdot t]\!]$.

Rule T-Var-Base refines the type of a variable with its exact value. Rule T-Const types a constant c using the function $ty(c)$ that is assumed to be sound, *i.e.* we assume that for each constant c, $c \in [\![ty(c)]\!]$. Rule T-Sub allows to weaken the type of a given expression by subtyping, discussed below. Rule T-If achieves path sensitivity by typing each branch under an environment strengthened with the value of the condition. Finally, the two let binding rules T-Let and T-Spec only differ in whether the type of the bound variable is inferred or taken from the syntax. Note that the last premise, a well-formedness condition, ensures that the bound variable does not escape (at the type level) the scope of the let form.

2. Subtyping: $\Gamma \vdash t_1 \preceq t_2$ *iff* $\forall \sigma \in [\![\Gamma]\!], e \in [\![\sigma \cdot t_1]\!].\ e \in [\![\sigma \cdot t_2]\!]$.

Rule S-Base uses the relation isValid($\cdot$) to check subtyping on basic types; we leave this relation abstract for now since we will refine it in the course of this section. Knowles and Flanagan [7] define subtyping between base refinements as:

$$\begin{aligned}
&\text{isValid}(\Gamma \vdash \{x{:}b \mid p_1\} \preceq \{x{:}b \mid p_2\}) \\
\text{iff} \quad &\forall \sigma \in [\![\Gamma, x{:}b]\!].\text{if } \models \sigma \cdot p_1 \text{ then } \models \sigma \cdot p_2
\end{aligned}$$

This definition makes checking undecidable, as it quantifies over all substitutions. We come back to decidability below.

3. Well-Formedness: Rule W-Base overloads isValid($\cdot$) to refer to well-formedness on base refinements. A base refinement $\{x{:}b \mid p\}$ is well-formed only when p is typed as a boolean:

$$\text{isValid}(\Gamma \vdash \{x{:}b \mid p\}) \textit{ iff } \Gamma, x{:}b \vdash p{:}\texttt{bool}$$

Inference In addition to being undecidable, the typing rules in Figure 2 are not syntax directed: several types do not come from the syntax of the program, but have to be guessed—they are colored in blue in Figure 2. These are: the argument type of a function (Rule T-Fun), the common (least upper bound) type of the branches of a conditional (Rule T-If), and the resulting type of let expressions (Rules T-Let and T-Spec), which needs to be weakened to not refer to variable x in order to be well-formed. Thus, to turn the typing relation into a type checking algorithm, one needs to address both decidability of subtyping judgments and inference of the aforementioned types.

3.2 Liquid Types

Liquid types [16] provide a decidable and efficient inference algorithm for the typing relation of Figure 2. For decidability,

```
Infer :: Env → Expr → Quals → Maybe Type
Infer Γ̂ ê ℚ = A <*> t̂
  where A = Solve C A0 and (t̂, C) = Cons Γ̂ ê
Cons  :: Env → Expr → (Maybe Type, [Cons])
Solve :: [Cons] → Sol → Maybe Sol
```

Figure 3. Liquid Inference Algorithm (`Cons` and `Solve` are defined in [9]).

the key idea of liquid types is to restrict refinement predicates to be drawn from a *finite* set of *predefined*, SMT-decidable predicates $q \in \mathbb{Q}$.

Syntax The syntax of *liquid predicates*, written $\hat{p}$, is:

$$
\begin{array}{rcll}
\hat{p} & ::= & \mathtt{true} & \text{True} \\
& | & q & \text{Predicate, with } q \in \mathbb{Q} \\
& | & \hat{p} \wedge \hat{p} & \text{Conjunction} \\
& | & \kappa & \text{Liquid Variable} \\
A & ::= & \cdot \mid A, \kappa \mapsto \overline{q} & \textit{Solution}
\end{array}
$$

A liquid predicate can be true (`true`), an element from the predefined set of predicates (q), a conjunction of predicates ($\hat{p} \wedge \hat{p}$), or a predicate variable (κ), called a *liquid variable*. A *solution* A is a mapping from liquid variables to a set of elements of $\mathbb{Q}$. The set $\overline{q}$ represents a variable-free liquid predicate using `true` for the empty set and conjunction to combine the elements otherwise.

Checking When all the predicates in $\mathbb{Q}$ belong to SMT-decidable theories, validity checking of λ_R^e: isValid($\Gamma \vdash \{x{:}b \mid p_1\} \preceq \{x{:}b \mid p_2\}$) which quantifies over all embeddings of the typing environment, can be SMT automated in a sound and complete way. Concretely, a subtyping judgment $\Gamma \vdash \{x{:}b \mid \hat{p}_1\} \preceq \{x{:}b \mid \hat{p}_2\}$ is valid *iff* under all the assumptions of Γ, the predicate $\hat{p}_1$ implies the predicate $\hat{p}_2$.

$$
\begin{gathered}
\text{isValid}(\Gamma \vdash \{x{:}b \mid \hat{p}_1\} \preceq \{x{:}b \mid \hat{p}_2\}) \\
\textit{iff} \\
\text{isSMTValid}(\textstyle\bigwedge\{\hat{p} \mid x{:}\{x{:}b \mid \hat{p}\} \in \Gamma\} \Rightarrow \hat{p}_1 \Rightarrow \hat{p}_2)
\end{gathered}
$$

Inference The liquid inference algorithm, defined in Figure 3, first applies the rules of Figure 2 using liquid variables as the refinements of the types that need to be inferred and then uses an iterative algorithm to solve the liquid variable as a subset of $\mathbb{Q}$ [16] (steps 1 and 2 of § 2.2).

More precisely, given a typing environment $\hat{\Gamma}$, an expression $\hat{e}$, and the fixed set of predicates $\mathbb{Q}$, the function `Infer` $\hat{\Gamma}$ $\hat{e}$ $\mathbb{Q}$ returns the type of the expression $\hat{e}$ under the environment $\hat{\Gamma}$, if it exists, or nothing otherwise. It first generates a template type $\hat{t}$ and a set of constraints `C` that contain liquid variables in the types to be inferred. Then it generates a solution A that satisfies all the constraints in `C`. Finally, it returns the type $\hat{t}$ in which all the liquid variables have been substituted by concrete refinements from the mapping in A.

The function `Cons` $\hat{\Gamma}$ $\hat{e}$ uses the typing rules in Figure 2 to generate the template type `Just` $\hat{t}$ of the expression $\hat{e}$, *i.e.* a type that potentially contains liquid variables, and the basic constraints that appear in the leaves of the derivation tree of the judgment $\hat{\Gamma} \vdash \hat{e}{:}\hat{t}$. If the derivation rules fail, then `Cons` $\hat{\Gamma}$ $\hat{e}$ returns `Nothing` and an empty constraint list. The function `Solve` C A uses the decidable validity checking to iteratively pick a constraint in $c \in C$ that is not satisfied, while such a constraint exists, and weakens the solution A so that c is satisfied. The function A `<*>` $\hat{t}$ applies the solution A to the type $\hat{t}$, if both contain `Just` values, otherwise returns `Nothing`. Here and in the following, we pose: $A_0 \doteq \lambda\kappa.\mathbb{Q}$.

The algorithm `Infer` $\hat{\Gamma}$ $\hat{e}$ $\mathbb{Q}$ is proved to be terminating and sound and complete with respect to the typing relation $\hat{\Gamma} \vdash \hat{e}{:}\hat{t}$ as long as all the predicates are conjunctions of predicates drawn from the set $\mathbb{Q}$.

3.3 Gradual Refinement Types

Gradual refinement types [8] extend the refinements of λ_R^e to include imprecise refinements like $x > 0 \wedge ?$. While they describe the static and dynamic semantics of gradual refinements, inference is left as an open challenge. Our work extends liquid inference to gradual refinements, therefore we hereby summarize their basics.

Syntax The syntax of gradual predicates in $\lambda_G^{\tilde{p}}$ is

$$
\begin{array}{rcll}
\tilde{p} & ::= & p & \text{Precise Predicate} \\
& | & p \wedge ? & \text{Imprecise Predicates, where } p \text{ is local}
\end{array}
$$

A predicate is either *precise* or *imprecise*. The syntax of an imprecise predicate $p \wedge ?$ allows for a *static part* p. Intuitively, with the predicate $x > 0 \wedge ?$, x is statically (and definitely) positive, but the type system can optimistically assume stronger, non-contradictory requirements about x. To make this intuition precise and derive the complete static and dynamic semantics of gradual refinements, Lehmann and Tanter [8] follow the Abstracting Gradual Typing methodology (AGT) [5]. Following AGT, a gradual refinement type (resp. predicate) is given meaning by *concretization* to the set of static types (resp. predicates) it represents. Defining this concretization requires introducing two important notions.

Specificity First, we say that p_1 is *more specific* than p_2, written $p_1 \preceq p_2$, *iff* p_2 is true when p_1 is true:

$$p_1 \preceq p_2 \;\doteq\; \forall\sigma.\ \text{if } \models \sigma \cdot p_1 \text{ then } \models \sigma \cdot p_2$$

Locality Additionally, in order to prevent imprecise formulas from introducing contradictions—which would defeat the purpose of refinement checking—Lehmann and Tanter [8] identify the need for the static part of an imprecise refinement to be *local*. Using an explicit syntax $p(x)$ to explicitly declare the variable x refined by the predicate p, a refinement is local if there exists a value v for which $p[v/x]$ is true; and this, for any (well-typed) substitution that closes the predicate. Formally:

$$\text{isLocal}(p(x)) \;\doteq\; \forall\sigma, \exists v.\ \models \sigma \cdot p[v/x]$$

Concretization Armed with specificity and locality, Lehmann and Tanter [8] define the concretization function $\gamma(\cdot)$, which maps gradual predicates to the set of the static predicates they represent.

$$\begin{array}{rcl} \gamma(p(x)) & \doteq & \{p\} \\ \gamma((p \wedge ?)(x)) & \doteq & \{p' \mid p' \preceq p, \text{isLocal}(p'(x))\} \end{array}$$

A precise predicate concretizes to itself (singleton), while an imprecise predicate denotes all the local predicates more specific than its static part. This definition extends naturally to types and environments.

$$\begin{array}{rcl} \gamma(\{x{:}b \mid \tilde{p}\}) & \doteq & \{\{x{:}b \mid p\} \mid p \in \gamma(\tilde{p}(x))\} \\ \gamma(x{:}\tilde{t_x} \to \tilde{t}) & \doteq & \{x{:}t_x \to t \mid t_x \in \gamma(\tilde{t_x}), t \in \gamma(\tilde{t})\} \\ \gamma(\tilde{\Gamma}) & \doteq & \{\Gamma \mid x{:}t \in \Gamma \textit{ iff } x{:}\tilde{t} \in \tilde{\Gamma}, t \in \gamma(\tilde{t})\} \end{array}$$

The denotations of gradual refinement types are similar to those from § 3.1. The denotation of a base *imprecise* gradual refinement $\{x{:}b \mid p \wedge ?\}$ includes all (gradually-typed) expressions that satisfy at least p.

Type Checking Figure 2 is used "as is" to type gradual expressions $\tilde{\Gamma} \vdash \tilde{e}{:}\tilde{t}$, save for the fact that the validity predicate must be lifted to operate on gradual types. $\widetilde{\text{isValid}}(\cdot)$ holds if there exists a justification, by concretization, that the static judgment holds. Precisely:

$$\begin{array}{rl} \widetilde{\text{isValid}}(\tilde{\Gamma} \vdash \tilde{t_1} \preceq \tilde{t_2}) & \doteq \exists \Gamma \in \gamma(\tilde{\Gamma}), t_1 \in \gamma(\tilde{t_1}), t_2 \in \gamma(\tilde{t_2}). \\ & \quad \text{isValid}(\Gamma \vdash t_1 \preceq t_2) \\ \widetilde{\text{isValid}}(\tilde{\Gamma} \vdash \tilde{t}) & \doteq \exists \Gamma \in \gamma(\tilde{\Gamma}), t \in \gamma(\tilde{t}).\text{isValid}(\Gamma \vdash t) \end{array}$$

4 Gradual Liquid Types

We now address the combination of liquid type inference and gradual refinements. We extend the work of Lehmann and Tanter [8] by adapting the liquid type inference algorithm to the gradual setting. To do so, we apply the abstract interpretation approach of AGT [5] to lift the `Infer` function (defined in § 3.2) so that it operates on gradual liquid types.

Below is the syntax of predicates in $\lambda^{\check{p}}_{GL}$, a gradual liquid core language whose predicates are gradual predicates where the static part of an imprecise predicate is a liquid predicate, with the additional requirement that it is *local* (def. in § 3.3).

$$\begin{array}{lll} \check{p} ::= \hat{p} & & \text{Precise Liquid Predicate} \\ \quad \mid \hat{p} \wedge ? & & \text{Imprecise Liquid Predicate, where } \hat{p} \text{ is local} \end{array}$$

The elements of $\lambda^{\check{p}}_{GL}$ are both gradual and liquid; *i.e.* expressions $\check{e}$ could also be written as $\hat{\tilde{e}}$. Also, we write ? as a shortcut for the imprecise predicate `true ∧ ?`.

Our goal is to define $\check{\mathtt{Infer}}$ $\check{\Gamma}$ $\check{e}$ $\mathbb{Q}$ so that it returns a type $\check{t}$ such that $\check{\Gamma} \vdash \check{e}{:}\check{t}$. After deriving $\check{\mathtt{Infer}}$ using AGT (§ 4.1), we provide an algorithmic characterization of $\check{\mathtt{Infer}}$ (§ 4.2), which serves as the basis for our implementation. We present the properties that $\check{\mathtt{Infer}}$ satisfies in § 4.3.

4.1 Lifting Liquid Inference

We define the function $\check{\mathtt{Infer}}$ using the abstracting gradual typing methodology [5]. In general, AGT defines the consistent lifting of a function `f` as: $\tilde{\mathtt{f}}\ \tilde{t} = \alpha(\{\mathtt{f}\ t \mid t \in \gamma(\tilde{t})\})$, where α is the sound and optimal abstraction function that, together with γ, forms a Galois connection.

The question is how to apply this general approach to the liquid type inference algorithm. We answer this question via trial-and-error.

Try 1. Lifting `Infer` Assume we lift `Infer` in a similar manner, *i.e.* we pose

$$\check{\mathtt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q} = \alpha(\{\mathtt{Infer}\ \Gamma\ e\ \mathbb{Q} \mid \Gamma \in \gamma(\check{\Gamma}), e \in \gamma(\check{e})\})$$

This definition of $\check{\mathtt{Infer}}$ is too strict: it rejects expressions that should be accepted. Consider for instance the following expression $\check{e}$ that defines a function `f` with an imprecisely-refined argument:

```
// onlyPos :: {v:Int | 0 < v} → Int
// check :: Int → Bool
let f :: x:{Int | ?} → Int
    f x = if check x then onlyPos x else
        onlyPos (-x)
in f 42
```

There is no single static expression $e \in \gamma(\check{e})$ such that the definition of `f` above type checks; for any $\mathbb{Q}$ we will get `Infer` {} e $\mathbb{Q}$ = `Nothing` and abstracting the empty set denotes a type error. This behavior breaks the flexibility programmers expect from gradual refinements (this example is based on the motivation example of Lehmann and Tanter [8]). One expects that $\check{\mathtt{Infer}}$ {} $\check{e}$ $\mathbb{Q}$ should simply return `Int`. Note that this is the same reason that Garcia et al. [5] do not lift the typing relation as a whole, but instead lift type functions and predicates used in the definition of the typing relation. Here, as described in § 3.2, `Infer` calls the functions `Cons` and `Solve`, which in turn calls the function `isValid`. Which of these functions should we lift?

Try 2. Lifting `isValid` To our surprise, using gradual validity checking (the lifting of `isValid`, § 3.3) leads to an unsound inference algorithm! This is because, soundness of static inference implicitly relies on the property of validity checking that if two refinements p_1 and p_2 are right-hand-side valid, then so is their conjunction, *i.e.*:

$$\begin{array}{ll} \text{If} & \text{isValid}(\Gamma \vdash \{v{:}b \mid p\} \preceq \{v{:}b \mid p_1\}) \\ \text{and} & \text{isValid}(\Gamma \vdash \{v{:}b \mid p\} \preceq \{v{:}b \mid p_2\}), \\ \text{then} & \text{isValid}(\Gamma \vdash \{v{:}b \mid p\} \preceq p_1 \wedge p_2) \end{array}$$

But this property does not hold for gradual validity checking, because for any logical predicate q, it is true that $(q \Rightarrow p_1 \wedge q \Rightarrow p_1) \Rightarrow (q \Rightarrow p_1 \wedge p_2)$, but $((\exists q.(q \Rightarrow p_1)) \wedge (\exists q.(q \Rightarrow p_1))) \not\Rightarrow (\exists q.(q \Rightarrow p_1 \wedge p_2))$.

Try 3. Lifting `Solve` Let us try to lift `Solve`:
$\check{\mathtt{Solve}}\ A\ \check{C} = \{\mathtt{Solve}\ A\ C \mid C \in \gamma(\check{C})\}$

where $\check{C}$ denotes a gradual constraint (from Figure 1). This approach is successful and leads to a provably sound and complete inference algorithm (§ 4.3).

Note that in the definition of $\check{\texttt{Solve}}$, we do not appeal to abstraction. This is because we can directly define $\check{\texttt{Infer}}$ to consider all produced solutions instead.

$\check{\texttt{Infer}}$ :: $\check{\texttt{Env}} \rightarrow \check{\texttt{Expr}} \rightarrow$ Quals $\rightarrow$ Set Type
$\check{\texttt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q} = \{\check{t}' \mid \texttt{Just}\ \check{t}'\ \texttt{<-}\ A\ \texttt{<*>}\ \check{t},\ A \in \check{\texttt{Solve}}\ A_0\ \check{C}\}$
  where $(\check{t},\ \check{C}) = \texttt{Cons}\ \check{\Gamma}\ \check{e}$

First, function Cons derives the typing constraints $\check{C}$, and if successful, the template type $\check{t}$ (step 1 of § 2.4). Next, we use the lifted $\check{\texttt{Solve}}$ to concretize and solve all the derived constraints (steps 2 and 3 of § 2.4, *resp.*). By keeping track of the conretizations that return non-Nothing solutions, we derive the safe concretizations of § 2. Finally, we apply each solution to the template gradual type, yielding a set of inferred types. We do not explicitly abstract the set of inferred types back to a single gradual type. Adding abstraction by exploiting the abstraction function defined by Lehmann and Tanter [8] is left for future work. The applications discussed in § 6 and § 7 make explicit use of the inferred set in order to assist users in understanding errors and migrating programs.[1]

4.2 Algorithmic Concretization

To make $\check{\texttt{Infer}}$ algorithmic, we need an algorithmic concretization function of a set of constraints, $\gamma(\check{C})$.

Recall the concretization of gradual predicates: $\gamma((p \wedge ?)(x)) \doteq \{p' \mid p' \preceq p, \text{isLocal}(p'(x))\}$. In general, this function cannot be algorithmically computed, since it ranges over the infinite domain of predicates. In gradual liquid refinements, the domain of predicates is restricted to the powerset of the *finite* domain $\mathbb{Q}$. We define the algorithmic concretization function $\gamma_{\mathbb{Q}}(\check{p}(x))$ as the intersection of the powerset of the finite domain $\mathbb{Q}$ with the gradual concretization function.

$$\gamma_{\mathbb{Q}}(\check{p}(x)) \ \doteq\ 2^{\mathbb{Q}} \cap \gamma(\check{p}(x))$$

Concretization of gradual predicates reduces to (decidable) locality and specificity checking on the elements of $\mathbb{Q}$.

$$\gamma_{\mathbb{Q}}((\hat{p} \wedge ?)(x)) \ \doteq\ \{\hat{p}' \mid \hat{p}' \in 2^{\mathbb{Q}}, \hat{p}' \preceq \hat{p}, \text{isLocal}(\hat{p}'(x))\}$$

We naturally extend the algorithmic concretization function to typing environments, constraints, and list of constraints.

$$\begin{aligned}
\gamma_{\mathbb{Q}}(\check{\Gamma}) &\doteq \{\hat{\Gamma} \mid x{:}\hat{t} \in \hat{\Gamma} \textit{ iff } x{:}\check{t} \in \check{\Gamma}, \hat{t} \in \gamma_{\mathbb{Q}}(\check{t})\} \\
\gamma_{\mathbb{Q}}(\check{\Gamma} \vdash \check{t}_1 \preceq \check{t}_2) &\doteq \{\hat{\Gamma} \vdash \hat{t}_1 \preceq \hat{t}_2 \mid \hat{\Gamma} \in \gamma_{\mathbb{Q}}(\check{\Gamma}), \hat{t}_i \in \gamma_{\mathbb{Q}}(\check{t}_i)\} \\
\gamma_{\mathbb{Q}}(\check{\Gamma} \vdash \check{t}) &\doteq \{\hat{\Gamma} \vdash \hat{t} \mid \hat{\Gamma} \in \gamma_{\mathbb{Q}}(\check{\Gamma}), \hat{t} \in \gamma_{\mathbb{Q}}(\check{t}), \} \\
\gamma_{\mathbb{Q}}(\check{C}) &\doteq \{\hat{C} \mid c \in \hat{C} \textit{ iff } \check{c} \in \check{C}, \hat{c} \in \gamma_{\mathbb{Q}}(\check{c})\}
\end{aligned}$$

We use $\gamma_{\mathbb{Q}}(\cdot)$ to define an algorithmic version of $\check{\texttt{Solve}}$:
$\check{\texttt{Solve}}\ A\ \check{C} = \{\texttt{Solve}\ A\ \hat{C} \mid \hat{C} \in \gamma_{\mathbb{Q}}(\check{C})\}$

[1] In standard gradual typing, the set of static types denoted by a gradual type can be infinite, hence abstraction is definitely required. In contrast, here the structure of types is fixed, and the set of possible liquid refinements, even if potentially large, is finite. We can therefore do without abstraction. We discuss implementation considerations in § 6.

which in turn yields an algorithmic version of $\check{\texttt{Infer}}$.

4.3 Properties of Gradual Liquid Inference

We prove that the inference algorithm $\check{\texttt{Infer}}$ satisfies the the correctness criteria of Rondon et al. [16], as well as the static criteria for gradually-typed languages [20].[2] The corresponding proofs can be found in supplementary material [9].

4.3.1 Correctness of Inference

The algorithm $\check{\texttt{Infer}}$ is sound, complete, and terminates.

Theorem 4.1 (Correctness). *Let $\mathbb{Q}$ be a finite set of predicates from an SMT-decidable logic, $\check{\Gamma}$ a gradual liquid environment, and $\check{e}$ a gradual liquid expression. Then*

- ***Soundness*** *If $\check{t} \in \check{\texttt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$, then $\check{\Gamma} \vdash \check{e}{:}\check{t}$.*
- ***Completeness*** *If $\check{\texttt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q} = \emptyset$, then $\nexists \check{t}.\ \check{\Gamma} \vdash \check{e}{:}\check{t}$.*
- ***Termination*** *$\check{\texttt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$ terminates.*

We note that, unlike the Infer algorithm that provably returns the strongest possible solution, it is not clear how to relate the set of solutions returned by $\check{\texttt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$ with the rest of the types that satisfy $\check{\Gamma} \vdash \check{e}{:}\check{t}$.

4.3.2 Gradual Typing Criteria

Siek et al. [20] list three criteria for the static semantics of a gradual language, which the $\check{\texttt{Infer}}$ algorithm satisfies. These criteria require the gradual type system *(i)* is a conservative extension of the static type system, *(ii)* is flexible enough to accommodate the dynamic end of the typing spectrum (in our case, unrefined types), and *(iii)* supports a smooth connection between both ends of the spectrum.

(i) Conservative Extension The gradual inference algorithm $\check{\texttt{Infer}}$ coincides with the static algorithm Infer on terms that only rely on precise predicates. More specifically, if Infer infers a static type for a term, then $\check{\texttt{Infer}}$ returns only that type, for the same term. Conversely, if a term is not typeable with Infer, it is also not typeable with $\check{\texttt{Infer}}$.

Theorem 4.2 (Conservative Extension). *If $\texttt{Infer}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \texttt{Just}\ \hat{t}$, then $\check{\texttt{Infer}}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \{\hat{t}\}$. Otherwise, $\check{\texttt{Infer}}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \emptyset$.*

(ii) Embedding of imprecise terms We then prove that given a well-typed unrefined term (*i.e.* simply-typed), refining all base types with the unknown predicate ? yields a well-typed gradual term. This property captures the fact that it is possible to "import" a simply-typed term into the gradual liquid setting. (In contrast, this is not possible without gradual refinements: just putting true refinements to all base types does not yield a well-typed program.)

To state this theorem, we use t_s to denote simple types (b and $t_s \rightarrow t_s$) and similarly e_s and Γ_s for terms and environments. The simply-typed judgment is the standard one.

[2] Because this work focuses on the static semantics, *i.e.* the inference algorithm, we do not discuss the dynamic part of the gradual guarantee, which has been proven for gradual refinement types [8].

The $\lceil\cdot\rceil$ function turns simple types into gradual liquid types by introducing the unknown predicate on every base type (and naturally extended to environments and terms):

$$\lceil b \rceil = \{v{:}b \mid ?\} \qquad \lceil t_1 \rightarrow t_2 \rceil = x{:}\lceil t_1 \rceil \rightarrow \lceil t_2 \rceil$$

Theorem 4.3 (Embedding of Unrefined Terms). *If* $\Gamma_s \vdash e_s{:}t_s$*, then* $\texttt{Inf}\check{\texttt{e}}\texttt{r}\ \lceil \Gamma_s \rceil\ \lceil e_s \rceil\ \mathbb{Q} \neq \emptyset$.

(iii) Static Gradual Guarantee Finally, we prove the static *gradual guarantee*: typeability is monotonic in the precision of type information, *i.e.*, making type annotations less precise cannot introduce new type errors. We first define the notion of precision in terms of algorithmic concretization:

Definition 4.4 (Precision of Gradual Types). $\check{t}_1$ is less precise than $\check{t}_2$, written as $\check{t}_1 \sqsubseteq \check{t}_2$, *iff* $\gamma_{\mathbb{Q}}(\check{t}_1) \subseteq \gamma_{\mathbb{Q}}(\check{t}_2)$.

Precision naturally extends to type environments and terms.

Theorem 4.5 (Static Gradual Guarantee). *If* $\check{\Gamma}_1 \sqsubseteq \check{\Gamma}_2$ *and* $\check{e}_1 \sqsubseteq \check{e}_2$*, then for every* $\check{t}_{1i} \in \texttt{Inf}\check{\texttt{e}}\texttt{r}\ \check{\Gamma}_1\ \check{e}_1\ \mathbb{Q}$ *there exists* $\check{t}_{1i} \sqsubseteq \check{t}_{2i}$ *so that* $\check{t}_{2i} \in \texttt{Inf}\check{\texttt{e}}\texttt{r}\ \check{\Gamma}_2\ \check{t}_2\ \mathbb{Q}$.

For a given term and type environment, the theorem ensures that, for every inferred type, the algorithm infers a less precise type when run on a less precise term and environment.

5 Implementation

We implemented $\texttt{Inf}\check{\texttt{e}}\texttt{r}$ as `GuiLT`, an extension to Liquid Haskell [24] that takes a Haskell program annotated with gradual refinement specifications and returns an `.html` interactive file that lets the user explore all safe concretizations.

Concretely, `GuiLT` uses the existing API of Liquid Haskell to implement the three steps of gradual liquid type checking steps described in § 2.4 (and formalized in § 4): *1)* First, `GuiLT` calls the Liquid Haskell API to generate subtyping constraints that contain both liquid variables and imprecise predicates. *2)* Next, it calls the liquid API to collect all the refinement templates. The templates are used to map each occurrence of imprecise predicates in the constraints to a set of concretizations. These concretizations are combined to generate the possible concretizations of the constraints. *3)* Finally, using Liquid Haskell's constraint solving it decides the validity of each concretized constraint, while all the safe concretizations SCs are interactively presented to the user.

The implementation of `GuiLT` closely follows the theory of § 4, apart from the following practical adjustments.

Templates To generate the refinement templates we use Liquid Haskell's existing API. The generated templates consist of a predefined set of predicates for linear arithmetic ($v\ [< \mid \leq \mid > \mid \geq \mid = \mid \neq]\ x$), comparison with zero ($v\ [< \mid \leq \mid > \mid \geq \mid = \mid \neq]\ 0$) and length operations ($\text{len}\ v[\geq \mid >]0$, $\text{len}\ v = \text{len}\ x$, $v = \text{len}\ x$, $v = \text{len}\ x + 1$), where `v` and `x` respectively range over the refinement variable and any program variable. Application-specific templates are automatically abstracted from user-provided specifications, and the user can also explicitly define custom templates.

Depth For completeness of the theory, we check the validity of any solution, including all possible *conjunctions* of elements of $\mathbb{Q}$, which is not tractable in practice. The implementation uses an instantiation depth parameter that is 1 by default, meaning each ?? ranges over single templates. At depth 2, ?? ranges over conjunctions of (single) templates.

Sensibility Checking Each ?? can be instantiated with any templates that are local and specific. The implementation uses the SMT solver to check both. As an *optimization*, we perform a syntactic locality check to reject templates that are "non-sensible", for instance, syntactic contradictions of the form `x < v && v < x`. As a *heuristic*, we further filter out as non-sensible type-directed instantiations of the templates that, based on our experience, a user would not write, such as arithmetic operations on lists and booleans (*e.g.* `x < False`)—although potentially correct in Haskell through overloading.

Locality Checking To encode Haskell functions (*e.g.* `len`) in the refinements, Liquid Haskell is using uninterpreted SMT functions [26]. As Lehmann and Tanter [8] note, locality checking breaks under the presence of uninterpreted functions. For instance, the predicate `0 < len i` is not local on `i`, because `∃i. 0 < len i` is not SMT valid due to a model in which `len` is always negative. To check locality under uninterpreted functions we define a fresh variable (*e.g.* `leni`) for each function application (*e.g.* `len i`). For instance, `0 < len i` is local on `i` because under the new encoding `∃leni. 0 < leni` is SMT valid.

Partitions A critical optimization for efficiency is that after generation and before solving, the set of constraints is partitioned based on the constraint dependencies so that each partition is solved independently. Two constraints depend on each other when they contain the same liquid variable or when they contain different variables (*e.g.* $\texttt{k}_1$ and $\texttt{k}_2$) whose solutions depend on each other (*e.g.* $\texttt{x:\{k}_1\texttt{\}} \vdash \texttt{\{v|true\}}\preceq\texttt{\{v|k}_2\texttt{\}}$). That way, we reduce the number of ?? that appear in each set of independent constraints and thus the number of concretization combinations that need to be checked (which increases exponentially with the number of ?? accounting for all combinations of all concretizations).

6 Application I: Error Explanation

Next, we illustrate how `GuiLT` is used for error explanation. Consider the list indexing function:

```
(x:_) !!0 = x
(_:xs)!!i = xs!!(i - 1)
_     !!_ = error "Out of Bounds!"
```

Indexing is `0`-based and signals a runtime error for `length xs <= i`. Now consider a client indexing a list by its length:

```
client = [1, 2, 3] !! 3
```

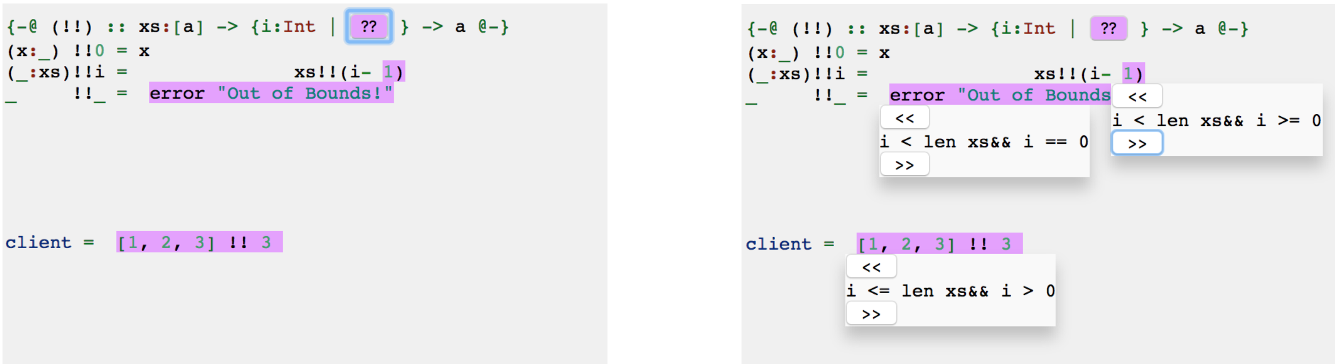

Figure 4. `GuiLT` GUI for error explanation of incompatible indexing.

Refinement types can be used to impose a false precondition on the `error` function and detect the out-of-bound crash. But when the two above incompatible definitions coexist is the error due to the definition or the client of indexing?

This question has no right answer in general. So we instead use gradual liquid types to explore possible resolutions of the error. To do so, we transform the statically ill-typed program into a gradually well-typed program by giving an imprecise refinement as a precondition to the indexing function:

```
(!!) :: xs:[a] → {i:Int | ?? } → a
```

Using `GuiLT`, we can explore resolutions to the original error.

Figure 4 was generated after running `GuiLT` on the above specification and code. The result of `GuiLT` is an `.html` file where each user specified ?? is turned into a colored button. Pressing a ?? button once (Left of Figure 4) highlights all of its uses (using different color for each ??). Pressing it again (Right of Figure 4) presents all safe concretizations to scroll through. In the indexing example, three gradual constraints are generated: *1)* for the recursive call of indexing, *2)* for the unreachable (due to the false precondition) error call, and *3)* for the client. Unsurprisingly, the the SCs for the recursive call and the client include the incompatible `0 <= i < len xs` and `0 < i <= len xs`, *resp.*. Interestingly, the unreachable constraint enjoys many SCs including `0 <= i <= len xs` and one given in the right of Figure 4, *i.e.* `i < len xs && i == 0`, since the case of indexing `0` from a non-empty list is covered in the first case of the indexing function.

The user can explore all SCs with the << and >> buttons. In the example, the three SCs are independent, but in many cases SCs can depend on each other (*e.g.* dependencies in function preconditions). In such cases, pressing a navigation button changes the values of all the dependent occurrences.

When the goal is to replace the ?? with a concrete refinement, going through all SCs can be overwhelming. In the example, there are 22, 10, and 13 SCs for the unreachable, client, and recursive calls, *resp.*. To accommodate the user, `GuiLT` generates an alternative interactive `.out.html` file by which the user can replace the ?? with any SC and observe the generated refinement errors. At the left of Figure 5 the ?? is replaced with the concrete refinement `0 <= i < len xs`, generating an error at the client site. Pressing >>, the user explores the next concrete refinement, at the right of Figure 5, where `0 < i <= len xs` generates an error in the recursive case of indexing. This exposes all the concretizations of ?? that render at least one constraint safe (here 22).

Quantitative Evaluation Table 1 summarizes a quantitative evaluation of the indexing example. We run `GuiLT` with instantiation depth (Depth) 1 and 2, *i.e.* the size of the conjunctions of the templates we consider (§ 5). The # ? column gives the number of imprecise refinements (as added by the user) and Occs gives the number each ?? appearing in the generated constraints. Column Cands denotes the candidate (*i.e.* well-typed) templates for each occurrence (§ 5). In the example, for depth 2, 68 templates are generated for each occurrence (*i.e.* [68, 68, 68] simplified as 68* for space). Then, `GuiLT` decides how many of the templates are sensible (Sens), local (Local), and specific (Spec); here, 38, 34, and 34, *resp.*. We note that all the local templates are specific, when the static part of the gradual refinement is true (*i.e.* `true && ??`, simplified as ??). Moreover, note that most non-local solutions were filtered out by the sensibility check. The constraints were split in 12 partitions (Parts), out of which 3 contained gradual refinements and had to be concretized. Each partition had 34 conretizations (# γ) out of which 22, 10, and 13 were safe concretizations (SCs), *resp.*. None of these conretizations was common for all the occurences of the ??, thus the `GuiLT` reports 0 static solutions (Sols), as expected. Finally, the running time of the whole process was 4.91 sec.

Next, we use these metrics to evaluate `GuiLT` on migrating three existing Haskell list libraries to Liquid Haskell.

7 Application II: Migration Assistance

As a second application, we use `GuiLT`'s error reporting GUI from § 6 to assist the migration of commonly used Haskell libraries to Liquid Haskell. Inter-language migration to strengthen type safety guarantees is one of the main motivations for gradual type systems [22]. Our study confirms

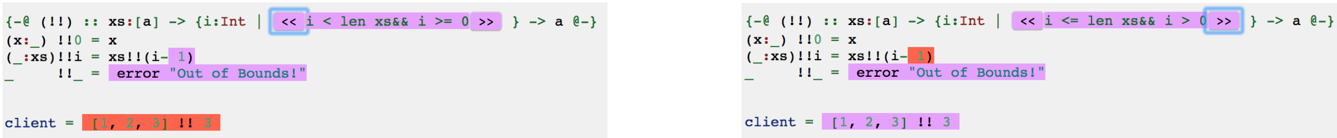

Figure 5. `GuiLT` GUI for liquid types exploration.

Depth	# ?	Occs	Cands	Sens	Local	Spec	Parts	# γ	SCs	Sols	Time (s)
1	1	[3]	[12*]	[11*]	[11*]	[11*]	3/12	11*	[8,0,6]	0	0.47
2	1	[3]	[68*]	[38*]	[34*]	[34*]	3/12	34*	[22,10,13]	0	4.91

Table 1. Quantitative Evaluation of the Indexing Example

that gradual liquid type inference can provide an effective bridge from traditional to refinement typed languages.

Benchmarks We used `GuiLT` to migrate three dependent, commonly used Haskell list libraries: *1)* `GHC.List`: provides commonly used list functions available at Haskell's `Prelude`, *2)* `Data.List`: defines more sophisticated list functions, *e.g.* list transposing, and *3)* `Data.List.NonEmpty`: lifts list functions to a non-empty list data type.

Migration Process A library consists of a set of function imports and definitions. Each function comes with its Haskell type and may be assigned to a gradual refinement type during migration. Migration is complete when, if possible, all functions are assigned to refinement types and type check under Liquid Haskell. The process proceeds in four steps:
Step 1: run Liquid Haskell to generate a set of type errors,
Step 2: fix the errors by inserting gradual refinements (`??`),
Step 3: use `GuiLT` to replace `??` with a generated SC, and
Step 4: go back to step 1 until no type errors are reported.

This process is iterative and interactive since refinement errors propagate between imported and client libraries and it is up to the user to decide how to resolve these errors.

Step 1: At the begging of the migration process the source files given to Liquid Haskell have no refinement type specifications. Still, there are two sources of refinement type errors: *1)* failure to satisfy imported functions preconditions, *e.g.* in § 6 the `error` function assumes the `false` precondition and *2)* incomplete patterns, *i.e.* a pattern-match that might fail at runtime, *e.g.* `scanr`'s result is matched to a non-empty list.

```
scanr _ q []     =  [q]
scanr f q (x:xs) =  f x q : qs
  where qs@(q:_) = scanr f q xs
```

Each time Step 1 is reiterated, type errors can get generated from function preconditions added in the next steps.

Step 2: The insertion of gradual refinements (`??`) is left to the user; they can add `??` either in the pre-conditions of defined functions or post-conditions of imported functions.

Step 3: Next, the user explores the generated SCs and chooses which one to replace `??` with. Often, the decision is trivial since only one SC coincides for all occurrences.

Step 4: Depending on the refinement inserted at Step 3, type errors might get generated in the current or imported libraries. If in Step 3 the user refined *(a)* the precondition of a function (*e.g.* `head` requires non-empty lists), then a type error might get generated at clients of the function both in- and outside the library, *(b)* the post-condition of the function (*e.g.* `scanr` always returns non-empty lists), then no error can get generated, *(c)* the post-condition of an imported functions upon which the function to be verified relies, then the imported function's specification may not be satisfied by its implementation. In this case the user can either assume the imported type (thus gradual verification) or update and re-check the imported library.

Evaluation Table 2 summarizes the migration case study; there are three subtables, one for each library: `GHC.List`, `Data.List`, and `Data.List.NonEmpty`. Within each of these tables, there is a row for every function which requires a refinement type to type check. Rows are collected into groupings of rounds, where round i lead to refinement type preconditions that trigger type errors in the functions of round $i + 1$. The columns of the table have the same meaning as that of Table 1; all results are for depth 1.

`GHC.List`: Liquid Haskell reported three static errors on the original version of `GHC.List`, *i.e.* with no user refinement type specifications. The function `errorEmp` is rejected as it is merely a wrapper around the `error` function; `scanr` and `scanr1` each have incomplete patterns assuming a non-empty list postcondition. `GuiLT` performed bad at all these three initial cases. It was unable to generate any SC for `errorEmp`, since the required false precondition is non local. It required more than one hour to generate SCs of `scanr` and we timed-out it in the case of `scanr1`. The reason for this is that `??` in post-conditions generated dependent set of constraints rendering our, crucial for efficiency, partition optimization (of 5) useless. Yet, `GuiLT` was useful in the next two rounds. Specification of `errorEmp` introduced errors in eight functions that were fixed using `GuiLT`: in seven cases the generated SCs coincide to exactly one predicate that was used to replace the `??`. One of this functions, `foldl` was

Rnd	Function	# ?	Occs	Cands	Sens	Local	Spec	Parts	# γ	SCs	Sols	Time (s)
`GHC.List` (56 functions defined and verified)												
1^{st}	`errorEmp`	1	[1]	[[5]]	[[4]]	[[4]]	[[4]]	1/4	[4]	[0]	0	1.00
	`scanr`	1	[4]	[6*]	[5*]	[5*]	[5*]	1/5	[625]	[125]	1	4640.20
	`scanr1`	2	[4,6]	[12*,5*]	[5*,4*]	[5*,4*]	[5*,4*]	1/5	[2560000]	??	??	timeout
2^{nd}	`head`	1	[1]	[[5]]	[[4]]	[[4]]	[[4]]	1/3	[4]	[1]	1	0.70
	`tail`	1	[1]	[[5]]	[[4]]	[[4]]	[[4]]	1/5	[4]	[1]	1	0.77
	`last`	1	[2]	[5*]	[4*]	[4*]	[4*]	2/4	4*	[4,1]	1	1.04
	`init`	1	[3]	[5*]	[4*]	[4*]	[4*]	2/8	[16,4]	[16,1]	1	3.12
	`foldl`	1	[3]	[5*]	[4*]	[4*]	[4*]	2/5	[4,16]	[1,16]	1	2.41
	`foldr1`	1	[1]	[[5]]	[[4]]	[[4]]	[[4]]	1/2	[4]	[1]	1	1.08
	`(!!)`	2	[4,4]	[5*,10*]	[4*,9*]	[4*,9*]	[4*,9*]	4/9	36*	[12,24,36,36]	6	7.81
	`cycle`	1	[2]	[5*]	[4*]	[4*]	[4*]	2/6	4*	[4,1]	1	1.37
3^{rd}	`maximum`	1	[3]	[5*]	[4*]	[4*]	[4*]	2/4	[4,16]	[1,16]	1	3.38
	`minimum`	1	[3]	[5*]	[4*]	[4*]	[4*]	2/4	[4,16]	[1,16]	1	2.80
`Data.List` (115 functions defined and verified)												
1^{st}	`maximumBy`	1	[3]	[5*]	[4*]	[4*]	[4*]	2/5	[16,4]	[16,1]	1	2.24
	`minimumBy`	1	[3]	[5*]	[4*]	[4*]	[4*]	2/5	[16,4]	[16,1]	1	2.40
	`transpose`	1	[3]	[12*]	[5*]	[5*]	[5*]	2/11	[25,5]	[0,4]	1	51.02
	`genericIndex`	2	[6,6]	[2*,5*]	[1*,4*]	[1*,4*]	[1*,4*]	6/12	4*	[3,4,4,1,1,4]	0	1.83
`Data.List.NonEmpty` (57 functions defined and verified)												
1^{st}	`fromList`	1	[2]	[5*]	[4*]	[4*]	[4*]	2/11	4*	[4,1]	1	1.41
	`cycle`	1	[1]	[[2]]	[[1]]	[[1]]	[[1]]	1/3	[1]	[0]	0	1.27
	`- toList`	2	[1,2]	[[2],5*]	[[1],4*]	[[1],4*]	[[1],4*]	2/3	4*	[1,3]	1	3.11
	`(!!)`	1	[4]	[10*]	[9*]	[9*]	[9*]	4/22	9*	[9,4,4,9]	1	5.96
2^{nd}	`cycle`	2	[1,1]	[[12],[5]]	[[5],[4]]	[[5],[4]]	[[5],[1]]	1/2	[5]	[2]	2	3.13
	`lift`	1	[1]	[[6]]	[[5]]	[[5]]	[[5]]	1/2	[5]	[2]	2	2.90
	`inits`	1	[1]	[[6]]	[[5]]	[[5]]	[[5]]	1/5	[5]	[2]	2	2.95
	`tails`	2	[1,1]	[[5],[6]]	[[4],[5]]	[[4],[5]]	[[4],[5]]	1/5	[20]	[6]	6	10.22
	`scanl`	1	[1]	[[6]]	[[5]]	[[5]]	[[5]]	1/5	[5]	[2]	2	2.23
	`scanl1`	1	[1]	[[6]]	[[5]]	[[5]]	[[5]]	1/2	[5]	[1]	1	1.13
	`insert`	1	[1]	[[12]]	[[5]]	[[5]]	[[5]]	1/5	[5]	[2]	2	2.25
	`transpose`	2	[1,1]	[[7],[12]]	[[6],[5]]	[[6],[5]]	[[6],[5]]	1/7	[30]	[6]	6	37.48
3^{rd}	`reverse`	2	[1,1]	[[5],[12]]	[[4],[5]]	[[4],[5]]	[[4],[5]]	2/3	[5,4]	[2,3]	5	0.96
	`sort`	2	[1,1]	[[12],[5]]	[[5],[4]]	[[5],[4]]	[[5],[4]]	2/3	[5,4]	[2,3]	5	1.03
	`sortBy`	2	[1,1]	[[12],[5]]	[[5],[4]]	[[5],[4]]	[[5],[4]]	2/3	[5,4]	[2,3]	5	0.97

Table 2. Evaluation of Migrations Assistance. Rnd: number of iterations to verify the function. Function: name of the function. # ? : number of ? inserted. Occs: times each ? is used. For each occurrence, we give the number of template candidates (Cands) and how many are sensible (Sens), local (Local), and specific (Spec). Parts: the number of partitions. For each partition, we give the number of concretizations (# γ) and safe concretizations (SCs). Sols: number of static solutions found. Time: time in sec.

further used by two more functions that were interactively migrated at the third and final specification round.

`Data.List`: Migration of `Data.List` only required one round. Four functions errored due to incomplete patterns or violation of preconditions of functions imported from previously verified `GHC.List`. Unsurprisingly, `GuiLT` was unable to find any SCs for `genericIndex`, a generic variant of `(!!)` that indexes lists using any integral (instead of integer) index, because it lacks arithmetic templates for integrals.

`Data.List.NonEmpty`: Migration was more interesting for the `Data.List.NonEmpty` library that manipulates the data type `NonEmpty a` of non-empty lists. The first round exposed that `fromList` requires the non-empty precondition. The `GHC.List` function `cycle` has a non-empty precondition, thus lifted to non-empty lists does not type check.

```
cycle :: NonEmpty a → NonEmpty a
cycle = fromList . List.cycle . toList
```

To migrate `cycle`, we first gradually refined the result type of `toList :: NonEmpty a → {xs:[a] | ??}` for which `GuiLT` suggested the single static refinement of `0 < len xs`.

Verification of non-empty list indexing calls requires invariants that relate the lengths of the empty and non-empty lists. Similarly, `GuiLT` finds SCs only after predicate templates that express such invariants are added. In general, to aid migration on user defined structures `GuiLT` requires the definition of domain specific templates.

On the second round, the non-empty precondition of `fromList` triggers errors to eight clients (*cf.* case *(a)*), all of which call functions that return (non-provably) non-empty lists. For example, `inits` lifts `List.inits` to non-empty lists.

```
inits = fromList . List.inits . toList
```

To migrate such functions, we add a gradual assumed specification for the `List` library function. For example

```
assume List.inits :: [a] → {o:[a] | ?? }
```

and use `GuiLT` to solve it to `0 < len o`, at which point, we could assume the imported specification or update and re-check the imported library (*cf.* case *(c)*).

The function `cycle` re-appears in the second round, due to the new precondition of `fromList`. Since the imported `List.cycle` already has a precondition `{i:[a] | 0 < len i}`, at this round we further strengthened the existing precondition with the imprecise refinement (where _ replaces the Haskell type for space):

```
assume List.cycle :: i:{_|0<len i && ??} → {o:_|??}
```

This is the only case in our experiments that we used a gradual refinement with a static part and thus the only case in which some local templates are rejected as non specific (here 3 out of 4 local templates are not specific).

Finally, we use `GuiLT` to derive higher-order specifications. The `lift` function lifts a list transformation to non-empty lists, `lift f = fromList . f . toList`, and comes with a comment that "If the provided function returns an empty list, this will raise an error.". Alerted by this comment, we use a ?? in higher-order position:

```
lift :: (i:[a] → {o:[b] | ??})
     → NonEmpty a → NonEmpty b
```

`GuiLT` produces two static solutions `0 < len o` and `len i == len o`. We choose the second, as the first implies list generation from empty lists. This leads to type errors in three clients, which are resolved in later rounds.

To sum up, `GuiLT` indeed is aiding Haskell to Liquid Haskell migration of real libraries, since the user needs merely to choose from the suggested SCs, instead of writing from scratch specifications. Many times, there is only one possible SC coinciding to all concretizations, thus the choice it trivial. When no suggestions are generated, *e.g.* `errorEmp`, the user falls back to the standard verification process. Currently, `GuiLT` requires a lot of user input (*e.g.* placing the ??), thus, it could be further automated.

8 Related Work

Liquid Types. Dependent types allow arbitrary expressions at the type level to express theorems on programs, while theorem proving is simplified by various automations ranging from tactics (*e.g.* Coq [1], Isabelle [28]) to external SMT solvers (*e.g.* F* [21]). Liquid types [16] restrict the expressiveness of the type specifications to decidable fragments of logic to achieve decidable type checking and inference.

Gradual Refinement Types. Several refinement type systems mix static verification with runtime checking. Hybrid types [7] use an external prover to statically verify subtyping when possible, otherwise a cast is automatically inserted to defer checking at runtime. Soft contract verification [11, 12, 23] works in the other direction, statically verifying contracts wherever possible, and otherwise leaving unverified contracts for checking at runtime. Ou et al. [13] allow the programmer to explicitly annotate whether an assertion is verified at compile- or runtime. Manifest contracts [6] formalize the metatheory of refinement typing in the presence of dynamic contract checking. Lehmann and Tanter [8] developed the first gradual refinement type system, which adheres to the refined criteria of Siek et al. [20]. None of these systems support inference; on the contrary, because refinements can be arbitrary, inference is impossible in these systems. Here we restrict gradual refinements to a finite set of predicates, in order to achieve inference by adaptation of the liquid inference procedure.

Gradual Type Inference. Many systems study type inference in presence of gradual types. Siek and Vachharajani [19] infer gradual types using unification, while Rastogi et al. [14] exploit type inference to improve the performance of gradually-typed programs. Like us, Garcia and Cimini [4] lift an inference algorithm from a core system to its gradual counterpart, by ignoring the unification constraints imposed by gradual types. In gradual liquid inference the constraints imposed by gradual refinements cannot be ignored, since unlike Hindley-Milner inference, the liquid algorithm starts from the strongest solution (*i.e.* false) for the liquid variables and uses constraints to iteratively weaken the solution.

Error Reporting. Inference algorithms are prone to misleading error messages [27], thus many algorithms have been proposed to improve feedback by using techniques like counter-example generation [10], heuristics [29], or learning [17]. Unlike these techniques, we uncover a novel application of gradual typing for error explanation, by observing concretizations of gradual types embed the inconsistencies that lead to refinement errors.

9 Conclusion

The concepts of gradual typing can be fruitfully exploited to assist in explaining type errors and migrating programs to a stronger typing discipline. We develop this intuition in the context of refinement types, yielding a novel integration of liquid type inference and gradual refinements. Gradual liquid type inference computes possible concretizations of unknown refinements in order for a program to be well-typed. In addition to developing the formal foundations, we provide an implementation integrated with Liquid Haskell, which hopefully will prove useful for migrating more Haskell libraries to the safer setting of Liquid Haskell.

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A Meta-Theory

We provide the full inference algorithm and prove the properties of § 4.3.

A.1 The Inference Algorithm

We define $\check{\texttt{Infer}}$ as in § 4.3:

$\check{\texttt{Infer}} :: \check{\texttt{Env}} \to \check{\texttt{Expr}} \to \texttt{Quals} \to [\check{\texttt{Type}}]$
$\check{\texttt{Infer}}\ \check{\Gamma}\ \check{e}\ \mathbb{Q} = \{\ \check{t}'\ \mid\ \texttt{Just}\ \check{t}'\ \texttt{<-}\ A\ \texttt{<*>}\ \check{t}$
$\qquad\qquad\qquad\qquad ,\ A \in \check{\texttt{Solve}}\ A_0\ \check{C}\ \mathbb{Q}\ \}$
$\quad \textbf{where}$
$\qquad A_0 = \lambda\kappa.\ \mathbb{Q}$
$\qquad (\check{t},\ \check{C}) = \texttt{Cons}\ \check{\Gamma}\ \check{e}$

$\check{\texttt{Solve}} :: \texttt{Sol} \to [\check{\texttt{Cons}}] \to \texttt{Quals} \to [\texttt{Maybe Sol}]$
$\check{\texttt{Solve}}\ A\ \check{C}\ \mathbb{Q} = \{\texttt{Solve}\ A\ \hat{C}\ \mid\ \hat{C} \in \gamma_{\mathbb{Q}}(\check{C})\}$

Note that unlike [16], for simplicity, we assume that the set of refinement predicates $\mathbb{Q}$ is not a set of templates, but an "instantiated" set of predicates. The algorithmic concretization on a list of constraints is defined as follows

$$
\begin{aligned}
\gamma_{\mathbb{Q}}(\check{p}(x)) &\doteq 2^{\mathbb{Q}} \cap \gamma(\check{p}(x))\\
\gamma_{\mathbb{Q}}(\check{\Gamma}) &\doteq \{\hat{\Gamma} \mid x{:}\hat{t} \in \hat{\Gamma}\ \mathit{iff}\ x{:}\check{t} \in \check{\Gamma}, \hat{t} \in \gamma_{\mathbb{Q}}(\check{t})\}\\
\gamma_{\mathbb{Q}}(\check{\Gamma} \vdash \check{t}_1 \preceq \check{t}_2) &\doteq \{\hat{\Gamma} \vdash \hat{t}_1 \preceq \hat{t}_2 \mid \hat{\Gamma} \in \gamma_{\mathbb{Q}}(\check{\Gamma}), \hat{t}_i \in \gamma_{\mathbb{Q}}(\check{t}_i)\}\\
\gamma_{\mathbb{Q}}(\check{\Gamma} \vdash \check{t}) &\doteq \{\hat{\Gamma} \vdash \hat{t} \mid \hat{\Gamma} \in \gamma_{\mathbb{Q}}(\check{\Gamma}), \hat{t} \in \gamma_{\mathbb{Q}}(\check{t}), \}\\
\gamma_{\mathbb{Q}}(\check{C}) &\doteq \{\hat{C} \mid c \in \hat{C}\ \mathit{iff}\ \check{c} \in \check{C}, \hat{c} \in \gamma_{\mathbb{Q}}(\check{c})\}
\end{aligned}
$$

We complete the definitions by repeating the `Solve` and `Cons` algorithms from [16].

The procedure `Solve` $A\ \hat{C}$ repeatedly weakens the solution A until the set of constraints $\hat{C}$ is satisfied, or returns `Nothing`.

$\texttt{Solve} :: \texttt{Sol} \to [\hat{\texttt{Cons}}] \to \texttt{Maybe Sol}$
$\texttt{Solve}\ A\ \hat{C} =$
$\quad \textbf{if}\ \texttt{exists}\ \hat{c} \in \hat{C}\ \texttt{s.t.}\ \neg\texttt{isValid}(A \cdot \hat{c})$
$\quad \textbf{then case}\ \texttt{Weaken}\ \hat{c}\ A\ \textbf{of}$
$\qquad\quad \texttt{Just}\ A' \to \texttt{Solve}\ A'\ \hat{C}$
$\qquad\quad \texttt{Nothing} \to \texttt{Nothing}$
$\quad \textbf{else}\ \texttt{Just}\ A$

$\texttt{Weaken} :: \hat{\texttt{Cons}} \to \texttt{Sol} \to \texttt{Maybe Sol}$
$\texttt{Weaken}\ (\hat{\Gamma} \vdash \{v{:}b \mid \theta \cdot \kappa\})\ A = \texttt{Just \$}$
$\quad \texttt{A}[\kappa \mapsto \{\texttt{q} \mid \texttt{q} \in A \cdot \kappa,\ \texttt{isValid}(A \cdot \hat{\Gamma} \vdash \{v{:}b \mid \theta \cdot q\})\}]$
$\texttt{Weaken}\ \hat{\Gamma} \vdash \{v{:}b \mid p\} \preceq \{v{:}b \mid \theta \cdot \kappa\}\ A = \texttt{Just \$}$
$\quad \texttt{A}[\kappa \mapsto \{\texttt{q} \mid \texttt{q} \in A \cdot \kappa,$
$\qquad \texttt{isValid}(A \cdot \hat{\Gamma} \vdash \{v{:}b \mid A \cdot p\} \preceq \{v{:}b \mid \theta \cdot q\})\}]$
$\texttt{Weaken}\ \_\ \_\ = \texttt{Nothing}$

The procedure `Cons` $\check{\Gamma}\ \check{e}$ is following the rules of Figure 2 to generate a template type of the expression $\check{e}$ and the set of constraints that should be satisfied.

$\texttt{Cons} :: \check{\texttt{Env}} \to \check{\texttt{Expr}} \to (\texttt{Maybe}\ \check{\texttt{Type}},\ [\check{\texttt{Cons}}])$
$\texttt{Cons}\ \check{\Gamma}\ \check{e} =$
$\quad \textbf{let}\ (\check{t}, \check{C}) = \texttt{Gen}\ \check{\Gamma}\ \check{e}$
$\quad (\check{t},\ \texttt{Split}\ \check{C})$

The procedure `Gen` $\check{\Gamma}\ \check{e}$ is using the typing rules of Figure 2 to generate a template type and a set of constraints.

$\texttt{Gen} :: \check{\texttt{Env}} \to \check{\texttt{Expr}} \to (\texttt{Maybe}\ \check{\texttt{Type}},\ [\check{\texttt{Cons}}])$
$\texttt{Gen}\ \check{\Gamma}\ x$
$\quad = \textbf{if}\ \check{\Gamma}(x) = \{v{:}b \mid \_\}$
$\qquad \textbf{then}\ (\texttt{Just}\ \tilde{R}vbv = x,\ \emptyset)$
$\qquad \textbf{else}\ (\texttt{Just}\ \check{\Gamma}(x),\ \emptyset)$
$\texttt{Gen}\ \check{\Gamma}\ c$
$\quad = (\texttt{Just}\ ty(c),\ \emptyset)$
$\texttt{Gen}\ \check{\Gamma}\ (\check{e}\ ::\ \check{t}) =$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_e,\ \check{C}) = \texttt{Gen}\ \check{\Gamma}\ \check{e}\ \textbf{in}$
$\quad (\texttt{Just}\ \check{t},\ (\check{\Gamma} \vdash \check{t}_e \preceq \check{t},\ \check{\Gamma} \vdash \check{t},\ \check{C}))$
$\texttt{Gen}\ \check{\Gamma}\ (\lambda x.\check{e}) =$
$\quad \textbf{let}\ \texttt{Just}\ x{:}\check{t}_x \to \check{t} = \texttt{Fresh}\ \check{\Gamma}\ \lambda x.\check{e}\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_e, \check{C}) = \texttt{Gen}\ (\check{\Gamma}, x{:}\check{t}_x)\ \check{e}\ \textbf{in}$
$\quad (\texttt{Just}\ (x{:}\check{t}_x \to \check{t}),\ (\check{\Gamma} \vdash \check{t}_e \preceq \check{t},\ \check{\Gamma} \vdash x{:}\check{t}_x \to \check{t}, \check{C}))$
$\texttt{Gen}\ \check{\Gamma}\ (\check{e}\ y) =$
$\quad \textbf{let}\ (\texttt{Just}\ (x{:}\check{t}_x \to \check{t}),\ \check{C}_1) = \texttt{Gen}\ \check{\Gamma}\ \check{e}\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_y, \check{C}_2) = \texttt{Gen}\ \check{\Gamma}\ y\ \textbf{in}$
$\quad (\texttt{Just}\ \check{t}[y/x], (\check{\Gamma} \vdash \check{t}_x \preceq \check{t}_y,\ \check{C}_1 \cup \check{C}_2))$
$\texttt{Gen}\ \check{\Gamma}\ \texttt{if}\ x\ \texttt{then}\ \check{e}_1\ \texttt{else}\ \check{e}_2) =$
$\quad \textbf{let}\ \texttt{Just}\ \check{t} = \texttt{Fresh}\ \check{\Gamma}\ \texttt{if}\ x\ \texttt{then}\ \check{e}_1\ \texttt{else}\ \check{e}_2\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_1, \check{C}_1) = \texttt{Gen}\ \check{\Gamma}, \_{:}\{v{:}\texttt{bool} \mid x\}\ \hat{e}_1\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_2, \check{C}_2) = \texttt{Gen}\ (\check{\Gamma}, \_{:}\{v{:}\texttt{bool} \mid \neg x\})\ \check{e}_2\ \textbf{in}$
$\quad (\texttt{Just}\ \check{t},\ (\check{\Gamma} \vdash \check{t},\ \check{\Gamma} \vdash \check{t}_1 \preceq \check{t},\ \check{\Gamma} \vdash \check{t}_2 \preceq \check{t}, \check{C}_1 \cup \check{C}_2))$
$\texttt{Gen}\ \check{\Gamma}\ (\texttt{let}\ x = \check{e}_x\ \texttt{in}\ \check{e}) =$
$\quad \textbf{let}\ \texttt{Just}\ \check{t} = \texttt{Fresh}\ \check{\Gamma}\ (\texttt{let}\ x = \check{e}_x\ \texttt{in}\ \check{e})\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_x, \check{C}_x) = \texttt{Gen}\ \check{\Gamma}\ \check{e}_x\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_e, \check{C}_e) = \texttt{Gen}\ (\check{\Gamma}, x{:}\check{t})\ \check{e}\ \textbf{in}$
$\quad (\texttt{Just}\ \check{t},\ (\check{\Gamma} \vdash \check{t},\ \check{\Gamma} \vdash \check{t}_e \preceq \check{t},\ \check{C}_x \cup \check{C}_e))$
$\texttt{Gen}\ \check{\Gamma}\ (\texttt{let}\ x{:}\check{t}_x = \check{e}_x\ \texttt{in}\ \check{e}) =$
$\quad \textbf{let}\ \texttt{Just}\ \check{t} = \texttt{Fresh}\ \check{\Gamma}\ (\texttt{let}\ x = \check{e}_x\ \texttt{in}\ \check{e})\ \textbf{in}$
$\quad \textbf{let}\ (\texttt{Just}\ \check{t}_e, \check{C}_e) = \texttt{Gen}\ (\check{\Gamma}, x{:}\check{t})\ \check{e}\ \textbf{in}$
$\quad (\texttt{Just}\ \check{t},\ (\check{\Gamma} \vdash \check{t}_x,\ \check{\Gamma} \vdash \check{t},\ \check{\Gamma} \vdash \check{t}_e \preceq \check{t},\ \check{C}_x \cup \check{C}_e))$
$\texttt{Gen}\ \_\ \_\ =$
$\quad (\texttt{Nothing},\ \emptyset)$

$\texttt{Fresh} :: \check{\texttt{Env}} \to \check{\texttt{Expr}} \to \texttt{Maybe}\ \check{\texttt{Type}}$
$\texttt{Fresh}\ \check{\Gamma}\ \check{e} = \texttt{HM}\ \textbf{type}\ \texttt{inference}$

The procedure `Gen` is using `Fresh`, the Hyndler Miller, unrefined type inference algorithm to generate liquid type templates with fresh refinement variables for the unknown types.

Finally, `Split C` using the well-formedness and sub-typing rules of Figure 2 to split the constraints into basic constraints.

$\texttt{Split} :: [\check{\texttt{Cons}}] \to\ [\check{\texttt{Cons}}]$
$\texttt{Split}\ \emptyset$
$\quad = \emptyset$
$\texttt{Split}\ (\check{\Gamma} \vdash \{v{:}b \mid p\}, \texttt{C})$
$\quad = (\check{\Gamma} \vdash \{v{:}b \mid p\},\ \texttt{Split C})$
$\texttt{Split}\ (\check{\Gamma} \vdash x{:}\check{t}_x \to \check{t}, \texttt{C})$
$\quad = \texttt{Split}\ (\check{\Gamma} \vdash \check{t}_x,\ \check{\Gamma}, x{:}\check{t}_x \vdash \check{t}, \texttt{C})$
$\texttt{Split}\ (\check{\Gamma} \vdash \{v{:}b \mid p_1\} \preceq \{v{:}b \mid p_2\},\ \texttt{C})$
$\quad = (\check{\Gamma} \vdash \{v{:}b \mid p_1\} \preceq \{v{:}b \mid p_2\},\ \texttt{Split C})$

$$\begin{aligned}&\mathtt{Split}\ (\check{\Gamma} \vdash x{:}\check{t}_{x1} \to \check{t}_1 \preceq x{:}\check{t}_{x2} \to \check{t}_2,\ \mathtt{C})\\ &\quad = \mathtt{Split}\ (\check{\Gamma} \vdash \check{t}_2 \preceq \check{t}_1,\ \check{\Gamma}, x{:}\check{t}_{x2} \vdash \check{t}_1 \preceq \check{t}_2,\ \mathtt{C})\end{aligned}$$

A.2 Correctness of Inference

Next, we prove Theorem 4.1. Let $\mathbb{Q}$ be a finite set of predicates from SMT-decidable logic, $\check{\Gamma}$be a gradual liquid environment, and $\check{e}$be a gradual liquid expression.

The proofs rely on the properties of the functions `Solve` and `Cons`. Since these two functions operate on liquid types, and are ignorant of the gradual setting, we directly port the proofs from [15].

Lemma A.1 (Constraint Generation). *Let* $(\mathtt{Just}\ \check{t}, \check{C}) = \mathtt{Cons}\ \check{\Gamma}\check{e}$. $\check{\Gamma} \vdash \check{e}{:}\check{t}'$ *iff there exists* A *so that* $\check{t}' \equiv A \cdot \check{t}$ *and* $\widetilde{isValid}(A \cdot \check{C})$.

Proof. Following Theorem 4 of Appendix A of [15]. Since `Cons` is just the algorithmic version of the rules of Figure 2 the theorem holds for any refinement type system with refinement variables, when the isValid$(\cdot)$ relation is exactly the same as in the premises of the rules in Figure 2. □

Lemma A.2 (Constraint Solving). *For every set of constraints* $\hat{C}$ *and qualifiers* $\mathbb{Q}$,

1. $\mathtt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C}$ *terminates.*
2. *If* $\mathtt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C} = \mathtt{Just}\ A$ *then* $isValid(A \cdot \hat{C})$.
3. *If* $\mathtt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C} = \mathtt{Nothing}$ *then* $\hat{C}$ *has no solution on* $\mathbb{Q}$.

Proof. Theorem 6 of Appendix A of [15]. □

Lemma A.3 (Gradual Validity).

1. $\widetilde{isValid}(A \cdot \check{c})$ *iff* $\exists \hat{c} \in \gamma_{\mathbb{Q}}(\check{c}).isValid(A \cdot \check{c})$
2. $\widetilde{isValid}(A \cdot \check{C})$ *iff* $\exists \hat{C} \in \gamma_{\mathbb{Q}}(\check{C}).isValid(A \cdot \check{C})$

Proof.

1. By case analysis on the shape of the constraint:
   - $\check{c} \equiv \check{\Gamma} \vdash \check{t}_1 \preceq \check{t}_2$.

$$\begin{gathered}\widetilde{\text{isValid}}(A \cdot \check{\Gamma} \vdash A \cdot \check{t}_1 \preceq A \cdot \check{t}_2)\\ \Leftrightarrow\\ \exists \hat{\Gamma} \in \gamma_{\mathbb{Q}}(\check{\Gamma}), \hat{t}_i \in \gamma_{\mathbb{Q}}(\check{t}_i).\text{isValid}(A \cdot \hat{\Gamma} \vdash A \cdot \hat{t}_1 \preceq A \cdot \hat{t}_2)\\ \Leftrightarrow\\ \exists \hat{c} \in \gamma_{\mathbb{Q}}(\check{c}).\text{isValid}(A \cdot \hat{c})\end{gathered}$$

   - $\check{c} \equiv \check{\Gamma} \vdash \check{t}$.

$$\begin{gathered}\widetilde{\text{isValid}}(A \cdot \check{\Gamma} \vdash A \cdot \check{t})\\ \Leftrightarrow\\ \exists \hat{\Gamma} \in \gamma_{\mathbb{Q}}(\check{\Gamma}), \hat{t} \in \gamma_{\mathbb{Q}}(\check{t}).\text{isValid}(A \cdot \hat{\Gamma} \vdash A \cdot \hat{t})\\ \Leftrightarrow\\ \exists \hat{c} \in \gamma_{\mathbb{Q}}(\check{c}).\text{isValid}(A \cdot \hat{c})\end{gathered}$$

2. By the definition of $\widetilde{\text{isValid}}(\cdot)$ and concretization of list of constraints.

$$\begin{gathered}\widetilde{\text{isValid}}(A \cdot \check{C})\\ \Leftrightarrow\\ \forall \check{c} \in \check{C}.\widetilde{\text{isValid}}(A \cdot \check{c})\\ \Leftrightarrow\\ \forall \check{c} \in \check{C}.\exists \hat{c} \in \gamma_{\mathbb{Q}}(\check{c}).\text{isValid}(A \cdot \check{c})\\ \Leftrightarrow\\ \exists \hat{C} \in \gamma_{\mathbb{Q}}(\check{C}).\text{isValid}(A \cdot \check{C})\end{gathered}$$

□

Theorem A.4 (Soundness).
If $\check{t} \in \mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$, *then* $\check{\Gamma} \vdash \check{e}{:}\check{t}$.

Proof. Since $\check{t} \in \mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$ then $\exists A$ so that

(1) $\check{t} = A \cdot \check{t}'$
(2) $\mathtt{Just}\ A \in \mathtt{So\check{l}ve}\ (\lambda\kappa.\mathbb{Q})\ \check{C}\ \mathbb{Q}$
(3) $(\mathtt{Just}\ \check{t}', \check{C}) = \mathtt{Cons}\ \check{\Gamma}\ \check{e}$

From (2), $\exists \hat{C} \in \gamma_{\mathbb{Q}}(\check{C})$ so that

(4) $\mathtt{Just}\ A \in \mathtt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C}$

From (4) and Lemma A.2 we get

(5) $\text{isValid}(A \cdot \hat{C})$

By Lemma A.3 we get

(6) $\widetilde{\text{isValid}}(A \cdot \check{C})$

By (1), (3), and Theorem A.1,

$$\check{\Gamma} \vdash \check{e}{:}\check{t}$$

□

Theorem A.5 (Completeness).
If $\mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q} = \emptyset$, *then* $\nexists \check{t}.\ \check{\Gamma} \vdash \check{e}{:}\check{t}$.

Proof. Assume that there exists $\check{t}$ so that $\check{\Gamma} \vdash \check{e}{:}\check{t}$. Then, by Lemma A.1, there exists an A so that

(1) $(\mathtt{Just}\ \check{t}', \check{C}) = \mathtt{Cons}\ \check{\Gamma}\ \check{e}$
(2) $\check{t} = A \cdot \check{t}'$
(3) $\widetilde{\text{isValid}}(A \cdot \check{C})$

From (3) and Lemma A.3 $\exists \hat{C} \in \gamma_{\mathbb{Q}}(\check{C})$ so that

(4) $\text{isValid}(A \cdot \hat{C})$

From (4) and inverting 3 of Lemma A.2 we get

(5) $\mathtt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C} \neq \mathtt{Nothing}$

By the definition of $\mathtt{So\check{l}ve}$ we get

(6) $\exists A'.\mathtt{Just}\ A' \in \mathtt{So\check{l}ve}\ (\lambda\kappa.\mathbb{Q})\ \check{C}\ \mathbb{Q}$

By the definition of $\mathtt{In\check{f}er}$ we get

$$\mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q} \neq \emptyset$$

Since we reached a contradiction, there cannot exist $\check{t}$ so that $\check{\Gamma} \vdash \check{e}{:}\check{t}$. □

Theorem A.6 (Termination). $\mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$ *terminates.*

Proof. Since both the set of constraints $\check{C}$ and the set of refinement predicates $\mathbb{Q}$ are finite, the concretizations $\gamma_{\mathbb{Q}}(\check{C})$ are also finite. Thus, $\mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$ calls $\mathtt{Solve}\ \cdot\ \cdot$ finite times and by Theorem A.2 $\mathtt{Solve}\ \cdot\ \cdot$ terminates, thus so does $\mathtt{In\check{f}er}\ \check{\Gamma}\ \check{e}\ \mathbb{Q}$. □

A.3 Criteria for Gradual Typing

Finally we prove the static criteria for gradual typing.

(i) Conservative Extension

Theorem A.7 (Conservative Extension). *If* $\mathtt{Just}\ \hat{t} = \mathtt{Infer}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q}$, *then* $\mathtt{In\check{f}er}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \{\hat{t}\}$. *Otherwise,* $\mathtt{In\check{f}er}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \emptyset$.

Proof. If $\mathtt{Just}\ \hat{t} = \mathtt{Infer}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q}$, then there exists an A, so that

(1) $\hat{t} = A \cdot \hat{t}'$
(2) $\mathtt{Just}\ A = \mathtt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C}$
(3) $(\mathtt{Just}\ \hat{t}', \hat{C}) = \mathtt{Cons}\ \hat{\Gamma}\ \hat{e}$

Since the generated constraints $\hat{C}$ contain no ?, then $\{\hat{C}\} = \gamma_{\mathbb{Q}}(\hat{C})$.

Thus, $\mathtt{Sol\check{v}e}\ (\lambda\kappa.\mathbb{Q})\ \hat{C}\ \mathbb{Q} = \mathtt{Just}\ A$. So, $\mathtt{In\check{f}er}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \{\hat{t}\}$.

Otherwise, $\mathtt{Infer}\ \hat{\Gamma}\ \hat{e}\ \mathbb{Q} = \mathtt{Nothing}$, because of a failure either at constraint generation or at solving. In either case $\mathtt{In\check{f}er}$ will also return $\emptyset$. □

(ii) Embedding of Unrefined Terms

Definition A.8 (Unrefined Type & Terms). Unrefined types and terms represent base types and lambda calculus terms that are typed using HindleyMilner inference: $\lfloor\Gamma\rfloor \vdash \lfloor e\rfloor{:}\lfloor t\rfloor$.

$$\begin{aligned}
\lfloor\{v{:}b \mid p\}\rfloor &= b \\
\lfloor x{:}t \to t\rfloor &= \lfloor t_x\rfloor \to \lfloor t\rfloor
\end{aligned}$$

$$\begin{aligned}
\lfloor c\rfloor &= c \\
\lfloor \lambda x.e\rfloor &= \lambda x.\lfloor e\rfloor \\
\lfloor x\rfloor &= x \\
\lfloor e\ x\rfloor &= \lfloor e\rfloor\ x \\
\lfloor \mathtt{if}\ x\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2\rfloor &= \mathtt{if}\ x\ \mathtt{then}\ \lfloor e_1\rfloor\ \mathtt{else}\ \lfloor e_2\rfloor \\
\lfloor \mathtt{let}\ x = e_x\ \mathtt{in}\ e\rfloor &= \mathtt{let}\ x = \lfloor e_x\rfloor\ \mathtt{in}\ \lfloor e\rfloor \\
\lfloor \mathtt{let}\ x{:}t = e_x\ \mathtt{in}\ e\rfloor &= \mathtt{let}\ x{:}\lfloor t\rfloor = \lfloor e_x\rfloor\ \mathtt{in}\ \lfloor e\rfloor
\end{aligned}$$

Definition A.9 (Imprecise Types & Terms). Imprecise types are refined with only ?. Imprecise terms only use imprecise type annotations.

$$\begin{aligned}
\lceil\{v{:}b \mid p\}\rceil &= \{v{:}b \mid ?\} \\
\lceil x{:}t \to t\rceil &= x{:}\lceil t_x\rceil \to \lceil t\rceil
\end{aligned}$$

$$\begin{aligned}
\lceil c\rceil &= c' \text{where } c' = c \wedge ty(c') = \lceil ty(c)\rceil \\
\lceil \lambda x.e\rceil &= \lambda x.\lceil e\rceil \\
\lceil x\rceil &= x \\
\lceil e\ x\rceil &= \lceil e\rceil\ x \\
\lceil \mathtt{if}\ x\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2\rceil &= \mathtt{if}\ x\ \mathtt{then}\ \lceil e_1\rceil\ \mathtt{else}\ \lceil e_2\rceil \\
\lceil \mathtt{let}\ x = e_x\ \mathtt{in}\ e\rceil &= \mathtt{let}\ x = \lceil e_x\rceil\ \mathtt{in}\ \lceil e\rceil \\
\lceil \mathtt{let}\ x{:}t = e_x\ \mathtt{in}\ e\rceil &= \mathtt{let}\ x{:}\lceil t\rceil = \lceil e_x\rceil\ \mathtt{in}\ \lceil e\rceil
\end{aligned}$$

Lemma A.10 (Well formedness of imprecise types). $\Gamma \vdash \lceil t\rceil$

Proof. Trivial, since true always belongs to the concretization of imprecise gradual refinements. □

Lemma A.11 (Imprecise Subtyping). *If all of the refinements in t are local, then $\Gamma \vdash t \preceq \lceil t\rceil$ and $\Gamma \vdash \lceil t\rceil \preceq t$.*

Proof. By induction on t.

- $\Gamma \vdash \{v{:}b \mid p\} \preceq \{v{:}b \mid ?\}$, since $\{v{:}b \mid \mathtt{true}\} \in \gamma(\{v{:}b \mid ?\})$.
- $\Gamma \vdash \{v{:}b \mid ?\} \preceq \{v{:}b \mid p\}$, since $\{v{:}b \mid p\} \in \gamma(\{v{:}b \mid ?\})$.
- $\Gamma \vdash x{:}t_x \to t \preceq x{:}\lceil t_x\rceil \to \lceil t\rceil$, since $\Gamma \vdash \lceil t_x\rceil \preceq t_x$ and $\Gamma, x{:}\lceil t_x\rceil \vdash t \preceq \lceil t\rceil$ by inductive hypothesis.
- $\Gamma \vdash x{:}\lceil t_x\rceil \to \lceil t\rceil \preceq x{:}t_x \to t$, since $\Gamma \vdash t_x \preceq \lceil t_x\rceil$ and $\Gamma, x{:}t_x \vdash \lceil t\rceil \preceq t$ by inductive hypothesis.

□

Lemma A.12 (Imprecise Terms). *If all the refinements for constants and user types are local, then if $\lfloor\Gamma\rfloor \vdash \lfloor e\rfloor{:}\lfloor t\rfloor$, then $\lceil\Gamma\rceil \vdash \lceil e\rceil{:}\lceil t\rceil$.*

Proof. The proof proceeds by induction on the derivation tree of $\lfloor\Gamma\rfloor \vdash \lfloor e\rfloor{:}\lfloor t\rfloor$.

- $e \equiv x$. By assumption, $x \in \lceil\Gamma\rceil$.
  - If $\lceil\Gamma\rceil(x) = \{v{:}b \mid \_\}$, then $\lceil\Gamma\rceil \vdash x{:}\{v{:}b \mid v = x\}$. By Rule T-Sub and Lemmas A.11 and A.10 $\lceil\Gamma\rceil \vdash x{:}\{v{:}b \mid ?\}$.
  - Otherwise, $\lceil\Gamma\rceil \vdash x{:}\lceil\Gamma(x)\rceil$.
- $e \equiv \lambda x.e'$. By inversion of hypothesis $\lfloor\Gamma, x{:}t_x\rfloor \vdash \lfloor e'\rfloor{:}\lfloor t\rfloor$. By inductive hypothesis $\lceil\Gamma, x{:}t_x\rceil \vdash \lceil e'\rceil{:}\lceil t\rceil$. By Lemma A.10 $\lceil\Gamma\rceil \vdash \lceil x{:}t_x \to t\rceil$. So, by rule T-Fun $\lceil\Gamma\rceil \vdash \lceil e\rceil{:}\lceil x{:}t_x \to t\rceil$.
- $e \equiv e'\ x$. By inversion of hypothesis $\lfloor\Gamma\rfloor \vdash \lfloor e'\rfloor{:}\lfloor x{:}t_x \to t\rfloor$ and $\lfloor\Gamma\rfloor \vdash \lfloor x\rfloor{:}\lfloor t_x\rfloor$. By inductive hypothesis $\lceil\Gamma\rceil \vdash \lceil e'\rceil{:}\lceil x{:}t_x \to t\rceil$ and $\lceil\Gamma\rceil \vdash \lceil x\rceil{:}\lceil t_x\rceil$. By rule T-App $\lceil\Gamma\rceil \vdash \lceil e\rceil{:}\lceil t\rceil$.
- $e \equiv c$. By definition of $\lceil c\rceil$ this case falls in the previous.
- $e \equiv \mathtt{if}\ x\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2$. By inversion of the hypothesis $\lfloor\Gamma\rfloor \vdash x{:}\lfloor\{v{:}\mathtt{bool} \mid \_\}\rfloor$, $\lfloor\Gamma\rfloor \vdash \lfloor e_1\rfloor{:}\lfloor t\rfloor$, and $\lfloor\Gamma\rfloor \vdash \lfloor e_2\rfloor{:}\lfloor t\rfloor$. By inductive hypothesis $\lceil\Gamma\rceil \vdash x{:}\lceil\{v{:}\mathtt{bool} \mid \_\}\rceil$, $\lceil\Gamma\rceil \vdash \lceil e_1\rceil{:}\lceil t\rceil$, and $\lceil\Gamma\rceil \vdash \lceil e_2\rceil{:}\lceil t\rceil$. By weakening, Lemma A.10 and the rule T-If $\lceil\Gamma\rceil \vdash \lceil e\rceil{:}\lceil t\rceil$.
- $e \equiv \mathtt{let}\ x = e_x\ \mathtt{in}\ e'$ By inversion of the hypothesis $\lfloor\Gamma\rfloor \vdash \lfloor e_x\rfloor{:}\lfloor t_x\rfloor$ and $\lfloor\Gamma, x{:}t_x\rfloor \vdash \lfloor e'\rfloor{:}\lfloor t\rfloor$. By inductive hypothesis, rule T-Let, and Lemma A.10, $\lceil\Gamma\rceil \vdash \lceil e\rceil{:}\lceil t\rceil$.

- $e \equiv \texttt{let}\ x{:}t\ =\ e_x\ \texttt{in}\ e$ By inversion of the hypothesis $\lfloor\Gamma\rfloor \vdash \lfloor e_x\rfloor{:}\lfloor t'_x\rfloor$ and $\lfloor\Gamma, x{:}t_x\rfloor \vdash \lfloor e'\rfloor{:}\lfloor t\rfloor$. By hypothesis, Lemma A.11 and rule T-Sub $\lfloor\Gamma\rfloor \vdash \lfloor e_x\rfloor{:}\lfloor t_x\rfloor$. By inductive hypothesis, rule T-Spec, and Lemma A.10, $\lceil\Gamma\rceil \vdash \lceil e\rceil{:}\lceil t\rceil$.

$\square$

$\square$

Theorem A.13 (Embedding of Unrefined Terms). *If all the refinements in constants and user provided specifications are local, then if $\lfloor\Gamma\rfloor \vdash \lfloor e\rfloor{:}\lfloor t\rfloor$, then $\texttt{In}\check{\texttt{f}}\texttt{er}\ \lceil\Gamma\rceil\ \lceil e\rceil\ \mathbb{Q} \neq \emptyset$.*

Proof. Since by Lemma A.12 the theorem is proved by completeness of our inference algorithm, *i.e.* Theorem A.5. $\square$

(iii) Static Gradual Guarantee

Definition A.14 (Precision of Gradual Types). *$\check{t}_1 \sqsubseteq \check{t}_2$ iff $\gamma_{\mathbb{Q}}(\check{t}_1) \subseteq \gamma_{\mathbb{Q}}(\check{t}_2)$.*

Theorem A.15 (Static Gradual Guarantee). *If $\check{\Gamma}_1 \sqsubseteq \check{\Gamma}_2$ and $\check{e}_1 \sqsubseteq \check{e}_2$, then for every $\check{t}_{1i} \in \texttt{In}\check{\texttt{f}}\texttt{er}\ \check{\Gamma}_1\ \check{e}_1\ \mathbb{Q}$ there exists $\check{t}_{1i} \sqsubseteq \check{t}_{2i}$ so that $\check{t}_{2i} \in \texttt{In}\check{\texttt{f}}\texttt{er}\ \check{\Gamma}_2\ \check{t}_2\ \mathbb{Q}$.*

Proof. Since $\check{t}_{1i} \in \texttt{In}\check{\texttt{f}}\texttt{er}\ \check{\Gamma}_1\ \check{e}_1\ \mathbb{Q}$ then $\exists A$ so that

(1) $\check{t}_{1i} = A \cdot \check{t}_1$

(2) $\texttt{Just}\ A \in \texttt{So}\check{\texttt{l}}\texttt{ve}\ (\lambda\kappa.\mathbb{Q})\ \check{C}_1\ \mathbb{Q}$

(3) $(\texttt{Just}\ \check{t}_1, \check{C}_1) = \texttt{Cons}\ \check{\Gamma}_1\ \check{e}_1$

From (2),

(4) $\exists\hat{C}_1 \in \gamma_{\mathbb{Q}}(\check{C}_1).\texttt{Just}\ A \in \texttt{Solve}\ (\lambda\kappa.\mathbb{Q})\ \hat{C}_1$

Since Cons is preserves ?

(5) $(\texttt{Just}\ \check{t}_2, \check{C}_2) = \texttt{Cons}\ \check{\Gamma}_2\ \check{e}_2$

(6) $\check{t}_1 \sqsubseteq \check{t}_2$

(7) $\check{C}_1 \sqsubseteq \check{C}_2$

By (4) and (7) we get

(8) $\hat{C}_1 \in \gamma_{\mathbb{Q}}(\check{C}_2)$

So,

(9) $\texttt{Just}\ A \in \texttt{So}\check{\texttt{l}}\texttt{ve}\ (\lambda\kappa.\mathbb{Q})\ \check{C}_2\ \mathbb{Q}$

By (5) and (9) we get

(10) $A \cdot \check{t}_2 \in \texttt{In}\check{\texttt{f}}\texttt{er}\ \check{\Gamma}_2\ \check{e}_2\ \mathbb{Q}$

By (6), (10) and since A preserves ?

$$A \cdot \check{t}_1 \sqsubseteq A \cdot \check{t}_2$$